MODULE-2

PROPERTIES OF INTEGERS:-

Mathematical Induction, well ordering principle- mathematical induction, recursive relation, definitions fundamental principles of counting - the rules of sum and product, permutations, combinations - the binomial theorem, combinations with repetations.

MATHEMATICAL INDUCTION:

→ The method of MI is based on a principle called the induction principle. This principle can be proved using another principle known as well ordering principle.

→ These two principles highlight some important properties of integers.

Well ordering principle states that "Every non empty subset of Z^+ contains a smallest element". OR

"The set of all +ve integers is well ordered".

INDUCTION PRINCIPLE:

Induction principle states that, let $S(n)$ denotes an open statement that involves a +ve integer n. Suppose that the following conditions holds

(i) $s(1)$ is true

(ii) If $s(k)$ is true for some $k \in z^+$, then $s(k+1)$ is true.

Then, $s(n)$ is true for all $n \in z^+$.

METHOD OF MI:

To prove that a certain statement $s(n)$ is true for all z, $n \geq 1$.

The method of proving such a statement on the basis of Induction principle is called the method of MI.

This method consists of following 2 steps:

(i) Basic step: Verify the statement $s(1)$ is true i.e., verify that $s(n)$ is true for $n=1$.

(ii) Induction step: Assuming that $s(k)$ is true where $k \in z^+$, show that $s(k+1)$ is true.

PROBLEMS:

1) Prove by MI that, for all +ve integer $n \geq 1$,

$$1+2+3+ \cdots + n = \frac{1}{2} n(n+1).$$

$\Rightarrow$ Let $s(n): 1+2+3+ \cdots + n = \frac{1}{2} n(n+1)$

Basic step: We show that $s(1)$ is true

$$s(1) = \frac{1}{2}(1)(1+1)$$

$$s(1) = \frac{1}{2}(2)$$

$$1 = 1$$

$\therefore s(1)$ is true

Induction step: Assume that $s(k)$ is true for some $k \in z^+$

i.e; $S(k) = 1+2+3+ --- +k = \frac{1}{2} k(k+1)$

We show that $s(k+1)$ is true

Add $(k+1)$ on b.s

i.e; $1+2+3+ --- k+(k+1) = \frac{1}{2} k(k+1)+(k+1)$

$= (k+1)\left[\frac{1}{2}k+1\right]$

$= (k+1)\left[\frac{k+2}{2}\right]$

$= \frac{1}{2}(k+1)(k+2)$

$\therefore s(k+1)$ is true for some $k \in z^+$

Hence by MI, $s(n)$ is true for all $n \in z^+$.

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2) Prove that for each $n \in z^+$

A*A

$1^2+2^2+3^2+ ---- +n^2 = \frac{1}{6}n(n+1)(2n+1)$

$\Rightarrow$ Let $S(n)$: $1^2+2^2+3^2+ --- +n^2 = \frac{1}{6} n(n+1)(2n+1)$

We show that $s(1)$ is true B.S

$S(1) = 1^2 = \frac{1}{6}(1)(1+1)[2(1)+1]$

$1 = 1$

$\therefore S(1)$ is true

Induction step: Assume that $s(k)$ is true for some $k \in z^+$

i.e; $S(k) = 1^2+2^2+3^2+ ---- +k^2 = \frac{1}{6}k(k+1)(2k+1)$

We show that $s(k+1)$ is true

Add $(k+1)^2$ on b.s

i.e; $1^2+2^2+3^2+\dots+k^2+(k+1)^2 = \frac{1}{6}k(k+1)(2k+1)+(k+1)^2$

$= (k+1)\left[\frac{1}{6}k(2k+1)+(k+1)\right]$

$= (k+1)\left[\frac{k(2k+1)+6(k+1)}{6}\right]$

$= (k+1)\left[\frac{2k^2+k+6k+6}{6}\right]$

$= (k+1)\left[\frac{2k^2+7k+6}{6}\right]$

$= \frac{1}{6}(k+1)\left[2k^2+4k+3k+6\right]$

$= \frac{1}{6}(k+1)\left[2k(k+2)+3(k+2)\right]$

$= \frac{1}{6}(k+1)(2k+3)(k+2)$

$\therefore$ $S(k+1)$ is true for some $k \in \mathbb{Z}^+$

Hence by MI, $S(n)$ is true for all $\underline{\underline{n \in \mathbb{Z}^+}}$.

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3) Prove by M.I that,

mqp $1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

$\Rightarrow$ Let $S(n) = 1^2+3^2+5^2+\cdots+(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

Basic step: We show that $S(1)$ is true for $n \in Z^+$

$S(1)$: $1^2 = \frac{1}{3}(1)[2(1)-1][2(1)+1]$

$1 = \frac{1}{3}(1)(1)(3)$

$1 = 1$

$\therefore S(1)$ is true for $n \in Z^+$.

Induction step: Assume that $S(k)$ is true for some $k \in Z^+$

i.e; $S(k) : 1^2+3^2+5^2+\cdots+(2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

Now, we show that $S(k+1)$ is true for some $k \in Z^+$

Adding $(2k+1)^2$ on both sides,

$(2k-1)^2$

$[2(k+1)-1]^2$

$[2k+2-1]^2$

$[2k+1]^2$

$1^2+3^2+5^2+\cdots+(2k-1)^2+(2k+1)^2 = \frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$

$= (2k+1)\left[\frac{1}{3}k(2k-1)+\frac{(2k+1)}{3}\right]$

$= (2k+1)\left[\frac{k(2k-1)+3(2k+1)}{3}\right]$

$= \frac{(2k+1)}{3}[2k^2-k+6k+3] \Rightarrow \left[\frac{2k+1}{3}\right][2k^2+5k+3]$

$$\Rightarrow \frac{2k+1}{3}\left(2k^2+2k+3k+3\right)$$

$$\Rightarrow \frac{2k+1}{3}\left[2k(k+1)+3(k+1)\right]$$

$$\Rightarrow \frac{2k+1}{3}\left[(2k+3)(k+1)\right]$$

$$\Rightarrow \frac{1}{3}(k+1)(2k+1)(2k+3)$$

$\therefore$ $S(k+1)$ is true for some $k \in z^+$

Hence, by MI, $S(n)$ is true for all $n \in z^+$.

4) If n is any +ve integer, Prove that

$$1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$$

$\Rightarrow$ Let $S(n): 1\cdot 2+2\cdot 3+3\cdot 4+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.

Basic step: We show that $S(1)$ is true for $n \in z^+$

$$S(1): 1\cdot 2=\frac{1}{3}(1)(1+1)(1+2)$$

$$2=\frac{1}{3}(2)(3)$$

$$2=2$$

$\therefore$ $S(1)$ is true for $n \in z^+$

Induction step: Assume that $S(k)$ is true for some $k \in z^+$

i.e; $S(k): 1\cdot 2+2\cdot 3+3\cdot 4+\cdots+k(k+1)=\frac{1}{3}k(k+1)(k+2)$

$S(k): 1\cdot 2+2\cdot 3+3\cdot 4+\cdots+k(k+1)+$

k^2+3k+2 $(k+1)(k+2)=\frac{1}{3}k(k+1)(k+2)$

$+(k+1)(k+2)$

$$= (k+1)(k+2)\left[\frac{1}{3}k+1\right]$$

$$= (k+1)(k+2)\left[\frac{k+3}{3}\right]$$

$$= \frac{1}{3}(k+1)(k+2)(k+3)$$

$\therefore$ $S(k+1)$ is true for some $k \in z^+$

Hence by MI, $S(n)$ is true for all $n \in z^+$.

5) $1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6} n(n+1)(2n+7)$

map 2

Let $S(n)$: $1 \cdot 3 + 2 \cdot 4 + \cdots + n(n+2) = \frac{1}{6} n(n+1)(2n+7)$

Basic step: We show that $S(1)$ is true for $n \in z^+$

$$S(1) : 1 \cdot 3 = \frac{1 \times 2 \times 9}{6}$$

$$3 = 3 \text{ true}$$

$\therefore$ $S(1)$ is true for $n \in z^+$

Induction step: Assume that $S(k)$ is true for some $k \in z^+$

i.e., $S(k)$: $1 \cdot 3 + 2 \cdot 4 + \cdots + k(k+2) = \frac{1}{6} k(k+1)(2k+7) \rightarrow (1)$

Now, we show that $S(k+1)$ is true for some $k \in z^+$

Adding $(k+1)(k+3)$ on b.s of eqn (1), we get

$$1 \cdot 3 + 2 \cdot 4 + \cdots + (k+1)(k+3) = \frac{k(k+1)(2k+7)}{6} + (k+1)(k+3)$$

$$=\left(\frac{k+1}{6}\right)\left[k(2k+7)+6(k+3)\right]$$

$$=\frac{(k+1)}{6}\left[2k^2+13k+18\right]$$

$$=\frac{(k+1)(k+2)(2k+9)}{6}$$

$\therefore$ $S(k+1)$ is true for some $k \in \mathbb{Z}^+$

Hence, by MI, $S(n)$ is true for all $n \in \mathbb{Z}^+$.

Q) By MI, prove that $(n!) \geq 2^{n-1}$ for all integers $n \geq 1$

MPQ

⇒ Let $S(n): (n!) \geq 2^{n-1}$

Basic step: we show that $S(1)$ is true for $n \geq 1$

$$S(1): 1! \geq 2^{1-1}$$

$$1 \geq 2^0$$

$$1 \geq 1$$

∴ $S(1)$ is true for $n \geq 1$

Induction step: Assume that $S(k)$ is true for some $k \geq 1$

i.e., $S(k): (k!) \geq 2^{k-1}$ (or) $2^{k-1} \leq (k!)$

Now, we s.t $S(k+1)$ is true for some $k \geq 1$

$$(k+1)! \geq 2^{(k+1)-1}$$

$$2^{k-1} \leq (k!)$$

$$(k+1)! \geq 2^k \quad \text{(or)} \quad 2 \cdot 2^{k-1} \leq 2 \cdot k!$$

$$(k+1)! \geq 2 \cdot 2^{k-1} \quad\quad 2^k \leq (k+1)k! \; (\because 2 \leq k+1)$$

$$2^k \leq (k+1)! \quad\quad 2^k \leq (k+1)!$$

$$(k+1)! \geq 2^k \quad\quad (k+1)! \geq 2^k$$

∴ $S(k+1)$ is true for some $k \geq 1$

∴ Hence, $S(n)$ is true for all $n \geq 1$

7) Show that $4n < (n^2 - 7)$ $\forall$ positive integers $n \geq 6$.

$\Rightarrow$ Let $S(n)$: $4n < (n^2 - 7)$ $\forall$ $n \geq 6$

Basic step: we show that $S(6)$ is true

$$4(6) < (6^2 - 7)$$

$$24 < (36 - 7)$$

$$24 < 29$$

$\therefore$ It is clearly true

$\therefore$ $S(6)$ is true

Induction step: Assuming $S(k)$ is true for some $k \geq 6$

i.e., $S(k)$: $4k < (k^2 - 7)$

Now, we s.t $S(k+1)$ is true for some $k \geq 6$

$4k < (k^2 - 7)$

$4(k+1) < ((k+1)^2 - 7)$

$4k + 4 < (k^2 + 2k + 1 - 7)$

$4k + 4 < (k^2 + 2k - 6)$

Adding 4 on both sides

$$(4k + 4) < (k^2 - 7) + 4$$

$$4(k+1) < (k^2 - 7) + 4$$

$$4(k+1) < (k^2 - 7) + (2k + 1)$$

$$4(k+1) < (k^2 + 2k + 1) - 7$$

$$4(k+1) < (k+1)^2 - 7$$

$\therefore$ $S(k+1)$ is true for some $k \geq 6$

$\therefore$ Hence, $S(n)$ is true for all $\underline{n \geq 6}$

by MI.

8) Show that $2^n > n^2$ $\forall$ positive integers $n \geq 4$

$\Rightarrow$ Let $S(n)$: $2^n > n^2$ $\forall$ $n > 4$

Basic step: we show that $S(5)$ is true

$$2^5 > 5^2$$

$$32 > 25$$

It is clearly true

$\therefore S(5)$ is true

Induction step: Assuming $S(k)$ is true for some $k > 4$

i.e; $S(k)$: $2^k > k^2$

now, we show that $S(k+1)$ is true for some $k > 4$

Multiply 2 on b.s

$2^k > k^2$

$2^{k+1} > (k+1)^2$

$2^k \cdot 2 > (k^2+2k+1)$

$$2^k \cdot 2 > k^2 \cdot 2$$

$$2^{k+1} > 2 \cdot k^2$$

$$2^{k+1} > k^2 + k^2$$

$$2^{k+1} > k^2 + 2k + 1$$

($\because k^2 > 2k+1$ $\forall$ $k > 4$, $25 > 11$)

$$2^{k+1} > (k+1)^2$$

$\therefore S(k+1)$ is true for some $k > 4$

$\therefore$ Hence by MI, $S(n)$ is true for all $n > 4$

9) Prove that every positive integer $n \geq 24$ can be written as sum of 5's and / or 7's

⇒ Let $S(n)$: n can be written as sum of 5's and / or 7's, $n \geq 24$

Basic step: We show that $S(24)$ is true

$$24 \geq 7+7+5+5$$

$$24 \geq 24$$

It is clearly true

$\therefore S(24)$ is true

Induction step: Assuming $S(k)$ is true for some $k \geq 24$

i.e; $S(k)$: $k = (7+7+\cdots) + (5+5+5+\cdots)$

Now, we s.t $S(k+1)$ is true for some $k \geq 24$

Suppose, this representation of k has $\underline{r}$ no. of 7's and $\underline{s}$ no. of 5's. Since, $k \geq 24$ we should have $r \geq 2$ and $s \geq 2$

$k+1 = (7+7+\cdots) + (5+5+\cdots)$

$(5+5+\cdots)$

$r \geq 2 \quad s \geq 2$

$$k+1 = \underbrace{(7+7+\cdots)}_{r} + \underbrace{(5+5+\cdots)}_{s} + 1$$

$$k+1 = \underbrace{(7+7+\cdots)}_{r-2} + 7+7 + \underbrace{(5+5+\cdots)}_{s} + 1$$

$$k+1 = \underbrace{(7+7+\cdots)}_{r-2} + \underbrace{(5+5+\cdots)}_{s} + \underbrace{7+7+1}_{15}$$

$$k+1 = \underbrace{(7+7+\cdots)}_{r-2} + \underbrace{(5+5+\cdots)}_{s} + (5+5+5)$$

$$k+1 = \underbrace{(7+7+\cdots)}_{r-2} + \underbrace{(5+5+\cdots)}_{s+3}$$

$\therefore S(k+1)$ is true for some $k \geq 24$

$\therefore$ Hence by MI, $S(n)$ is true for all $n \geq 24$

10) Let $a_0 = 1$, $a_1 = 2$, $a_2 = 3$ & $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ for $n \geq 3$.

ma Prove that $a_n \leq 3^n$ $\forall$ +ve integers n $n \in z^+ (n \geq 1)$

$\Rightarrow$ Let $S(n)$: $a_n \leq 3^n$ $\forall$ $n \geq 1$

Basic step: We show that $S(0), S(1), S(2)$ is true

$n = 0$, $a_0 \leq 3^0 \Rightarrow 1 \leq 1$

$n = 1$, $a_1 \leq 3^1 \Rightarrow 2 \leq 3$

$n = 2$, $a_2 \leq 3^2 \Rightarrow 3 \leq 9$

$\therefore S(n)$ is true for $n = 0, 1, 2$

Induction step: Assuming that $S(k)$ is true for some $k \geq 3$

i.e; $S(k)$: $a_k \leq 3^k$ for some

Now, we show that $S(k+1)$ is true for some $k \geq 3$

Now, by using given data,

$a_k \leq 3^k$

$a_{k+1} \leq 3^{k+1}$

$a_{k+1} \leq 3^k \cdot 3$

$a_k \leq a_{k-1} + a_{k-2} + a_{k-3}$

$a_{k+1} \leq a_k + a_{k-1} + a_{k-2}$

$a_{k+1} \leq 3^k + 3^{k-1} + 3^{k-2}$ ($\because$ by our assumption)

$a_{k+1} \leq 3^k + 3^k + 3^k$ ($\because 3^{k-1} \leq 3^k$ & $3^{k-2} \leq 3^k$, $k \geq 3$)

$a_{k+1} \leq 3 \cdot 3^k$

$a_{k+1} \leq 3^{k+1}$

$\therefore S(k+1)$ is true for some $k \geq 3$

$\therefore$ Hence by MI, $S(n)$ is true for all $n \geq 1$

RECURSIVE:

Consider $a_1, a_2, ---, a_n$, a sequence in which a_1 is the first term, a_2 is the 2nd term ---- a_n is nth term. The element a_n is called the general term of the sequence denoted by $\{a_n\}$. If the first term is denoted by a_0, then nth term is denoted by a_{n-1} and their sequence is denoted by $\{a_{n-1}\}$.

For describing a sequence, 2 methods are commonly used:

(i) Explicit method (ii) Recursive method

i) In Explicit method, the general term of the sequence is explicitly indicated.

Ex: $E = \{a_n\}$, where $a_n = 2n$, $n \in Z^+$.

ii) In Recursive method, first few terms of the sequence are explicitly indicated and the general term is specified through a rule (formula) which indicates how to obtain new terms of the sequence from the terms already known.

Ex: Fibonacci numbers $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2} \ \forall n \geq 2$

NOTE:

1) A recursive definition of a sequence should contain 2 parts: in the 1st part, a few terms of the sequence are indicated explicitly and in the 2nd part, the rule obtain new terms for sequence is indicated.

2) Recursive rule / formula: is a formula for a_n in terms of a_{n-1}, a_{n-2} etc-- (or) a formula for a_{n+1} in terms of a_n, a_{n-1} etc--

PROBLEMS:

1) Obtain a recursive defn for the sequence $\{a_n\}$ in each of the following cases:

i) $a_n = 5n$

$\Rightarrow$ Given: $a_n = 5n$

when $n=1$, $a_1 = 5(1) = 5$

$n=2$, $a_2 = 5(2) = 10$

$n=3$, $a_3 = 5(3) = 15$

5, 10, 15

a_1, a_2, a_3

$a_2 = a_1 + 5$

$10 = 5 + 5$

$15 = 10 + 5$

$a_3 = a_2 + 5$ $\quad a_n + 5$

$\boxed{a_{n+1} = a_n + 5, n \geq 1}$

$a_n = a_{n-1} + 5, n > 1$

ii) $a_n = 6^n$

$\Rightarrow$ Given: $a_n = 6^n$

when $n=1$, $a_1 = 6$

$n=2$, $a_2 = 36$

$n=3$, $a_3 = 216$

$a_2 = 36 = a_1 \times 6 = 6 \times 6 = 36$

$a_3 = 216 = a_2 \times 6 = 36 \times 6 = 216$

$\boxed{a_{n+1} = a_n \cdot 6 , n \geq 1}$

iii) $a_n = n^2$ 3m

$\Rightarrow$ Given: $a_n = n^2$

when $n=1$, $a_1 = 1$

$n=2$, $a_2 = 4$

$n=3$, $a_3 = 9$

$a_2 = 4 \neq$

$a_3 = 9 \neq$

formula
even $\cdot$ 8n

$a_{n+1} =$

$2n+1 \to$ odd

$a_2 - a_1 = 4 - 1 = 3 = (2 \times 1) + 1 = 3$

$a_3 - a_2 = 9 - 4 = 5 = (2 \times 2) + 1 = 5$

$a_{n+1} - a_n = 2n+1$

$\boxed{a_{n+1} = a_n + 2n + 1 , n \geq 1}$

iv) $a_n = 2-(-1)^n$

$\Rightarrow$ Given: $a_n = 2-(-1)^n$

when $n=1$, $a_1 = 2-(-1)^1 = 2+1 = 3$

$n=2$, $a_2 = 2-(-1)^2 = 2-1 = 1$

$n=3$, $a_3 = 2-(-1)^3 = 2+1 = 3$, $a_n = 2-(-1)^n$

$$\boxed{a_{n+1} = 2-(-1)^{n+1}}$$

$$a_{n+1} - a_n = [2-(-1)^{n+1} - (2-(-1)^n)]$$

$$= [\cancel{2} - (-1)^{n+1} - \cancel{2} + (-1)^n]$$

$$a_{n+1} - a_n = [(-1)(-1)^n(-1)^1 + (-1)^n]$$

$$a_{n+1} - a_n = (-1)^n + (-1)^n$$

$$\boxed{a_{n+1} = 2(-1)^n + a_n , \ n \geq 1}$$

2) A sequence $\{c_n\}$ is defined recursively by $c_n = 3c_{n-1} - 2c_{n-2}$ $\forall$ $n \geq 3$, with $c_1 = 5$ and $c_2 = 3$ as a initial conditions. Show that $c_n = -2^n + 7$.

$\Rightarrow$ Given: $c_n = 3c_{n-1} - 2c_{n-2}$ — ① $\forall$ $n \geq 3$, $c_1 = 5$ and $c_2 = 3$

we show that $c_n = -2^n + 7$ for $n = 3$

$n = 3$, in eqn ①, — Basic step:

$$c_3 = 3c_2 - 2c_1$$

$$= 3(3) - 2(5)$$

$$= 9 - 10$$

$$\boxed{c_3 = -1}$$

$c_3 = -2^3 + 7$

It is true for $n=3$

$\therefore$ The given eqn is true

Induction step:- Assuming the given eqn is true for $n=k$

i.e; $c_k = -2^k + 7$, $k \geq \cancel{3}4$

$c_k = -2^k + 7$

$c_{k+1} = -2^{k+1} + 7$

$c_{k+1} = -2^k \cdot 2 + 7$

now, we show that given eqn is true for $n = k+1$

from eqn ①, $c_k = 3c_{k-1} - 2c_{k-2}$

$$c_{k+1} = 3c_k - 2c_{k-1}$$

$$c_{k+1} = 3(-2^k + 7) - 2(-2^{k-1} + 7)$$

$$c_{k+1} = -6^k + \cancel{7}21 + 4^{k-1} \cancel{+} - 14$$

$$c_{k+1} = -6^k + 21 - 2(-2^{k-1} + 7)$$

$$c_{k+1} = -6^k + 21 - 2(-2^k \cdot -2^{-1} + 7)$$

$$c_{k+1} = -6^k + 21 + 2^k - 14$$

$$c_{k+1} = -4^k + 7$$

$$c_{k+1} = -2^k \cdot 2 + 7$$

$$c_{k+1} = -2^{k+1} + 7$$

$$c_{k+1} = (3)(-2^k) + 21(-2)(-2^k(-2)^{-1} + 7)$$

$$= 3(-2^k) + 21(-2)^1(-2^k)(2)^{-1} - 14$$

$$= 3(-2^k) -$$

$$c_{k+1} = 3(-2^k+7) - 2(-2^{k-1}+7)$$

$(-2)^k = (-1)^k(-2)^k$

$-2^k = (-1)2^k$

$$= (-3)(2^k) + 21 + (-2)[-2^{k-1}] - 14$$

$$= (-3)2^k + (-1)(2)(-1)2^{k-1} + 7$$

$$= (-3)2^k + (2)^1(2)^{k-1} + 7$$

$$= (-3)2^k + 2^k + 7$$

$$= (-2)(2)^k + 7$$

$$= (-1)(2)^1(2)^k + 7$$

$$= (-1)2^{k+1} + 7$$

$$= -2^{k+1} + 7$$

3) The fibonacci numbers defined recursively by $F_0 = 0, F_1 = 1;$ $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Evaluate F_2 to F_{10}.

$\Rightarrow$ Given: $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$

$n = 2$, $F_2 = F_1 + F_0$

$F_2 = 1 + 0$

$F_2 = 1$

$n = 3$, $F_3 = F_2 + F_1$

$= 1 + 1$

$F_3 = 2$

$n = 4$, $F_4 = F_3 + F_2$

$= 2 + 1$

$F_4 = 3$

$n=5,\quad F_5 = F_4 + F_3$

$F_5 = 5$

$n=6,\quad F_6 = F_5 + F_4$

$= 8$

$n=7,\quad F_7 = F_6 + F_5$

$F_7 = 13$

$n=8,\quad F_8 = F_7 + F_6 = 21$

$n=9,\quad F_9 = F_8 + F_7 = 34$

$n=10,\quad F_{10} = F_9 + F_8 = \underline{\underline{55}}$

$F_{10} = 55$

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4) For the fibonacci sequence, $F_0, F_1, F_2 \ldots\ldots$ Prove that

vimp

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

⇒ Given: $F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

$n=0,\quad F_0 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0 - \left(\frac{1-\sqrt{5}}{2}\right)^0\right]$

$F_0 = \frac{1}{\sqrt{5}}(1-1)$

$F_0 = 0$

$n=1,\quad F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right]$

$F_1 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right)$

$F_1 = \frac{1}{\cancel{\sqrt{5}}}\left(\frac{\cancel{2}\cancel{\sqrt{5}}}{\cancel{2}}\right),\quad F_1 = 1$

$\therefore F_n$ is true for $n = 0, 1$

Assuming, F_n is true for $n = k$

i.e; $F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right]$

Now, we show that F_n is true for $n = k+1$

$$F_{k+1} = F_k + F_{k-1}$$

$$F_{k+1} = \frac{1}{\sqrt{5}} \left\{ \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - \left(\frac{1-\sqrt{5}}{2} \right)^k \right] + \left(\frac{1+\sqrt{5}}{2} \right)^{k-1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \right\}$$

$$\cancel{= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^k - 1-\sqrt{5} \right.}$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left[\frac{1+\sqrt{5}}{2} + 1 \right] - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left[\frac{1-\sqrt{5}}{2} + 1 \right] \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1+\sqrt{5}+2}{2} \right) - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left(\frac{1-\sqrt{5}+2}{2} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{3+\sqrt{5}}{2} \right) - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left(\frac{3-\sqrt{5}}{2} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{2(3+\sqrt{5})}{2 \cdot 2} \right) - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left(\frac{2(3-\sqrt{5})}{2 \cdot 2} \right) \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k-1} \left(\frac{1+\sqrt{5}}{2} \right)^2 - \left(\frac{1-\sqrt{5}}{2} \right)^{k-1} \left(\frac{1-\sqrt{5}}{2} \right)^2 \right]$$

$\because (1+\sqrt{5})^2$
$= 1 + 5 + 2\sqrt{5}$
$= 6 + 2\sqrt{5}$
$= 2(3+\sqrt{5})$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{k+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{k+1} \right]$$

$\therefore F_n$ is true for $n = k+1$

PRINCIPLE OF COUNTING:

The SUM RULE: Suppose two tasks $T1$ & $T2$ are to be performed if the task $T1$ can be performed in m different ways and the task $T2$ can be performed in n different ways and if these two tasks cannot be performed simultaneously, then one of the two tasks [$T1$ or $T2$] can be performed in $m+n$ ways.

Example: Suppose there are 16 boys and 18 girls in a class and we wish to select one of these student [either boy or girl] as CR.

The number of ways of selecting a boys is 16.

The number of ways of selecting a girl is 18.

Therefore, the number of ways of selecting a student [boy or girl] is $16+18 = 34$.

PRODUCT RULE:

Suppose that 2 tasks T_1 & T_2 are to be performed one after the other if T_1 can be performed in n_1 different ways, and each of these ways. There T_2 can be performed in n_2 different ways then both of the tasks can be performed in n_1 and n_2 ways.

Example: Suppose a person has 8 shirts and 5 ties, then he has $8 \times 5 = 40$ different ways of choosing a shirt and a tie.

PROBLEMS:

1) A license plate consist of 2 english letters followed by 4 digits if repetation are allowed, how many ways of the plates have only vowels [a, e, i, o, u] and event digits.

⇒ Each of the first two position in a plate can be filled in 5 ways [with vowels] and each of the remaining 4 places can be filled in 5 ways [with digits 0,2,4,6,8]

∴ The number of possible license plates of the given type is $(5 \times 5) \times (5 \times 5 \times 5 \times 5)$

$= 5^6$

$= 15625$

for not repeat

$(5 \times 4) \times (5 \times 4 \times 3 \times 2)$

2) Cars of a particular manufacturer come in 4 models, 12 colors, 3 engine sizes, and 2 transmission types

i) How many distinct cars [of these company can be manufactured]?

ii) Of these how many have the same colours.

⇒ i) By Product Rule, it follows that the number of distinct car can be manufactured =

$$4 \times 12 \times 3 \times 2 = 288$$

ii) For any choosen colour, the number of distinct car that can be manufactured is

$$= 4 \times 3 \times 2 = 24 \text{ cars}$$

3) A bit is either 0 or 1. A byte is a sequence of 8 bits

i) Find number of bits

ii) Number of bytes that begin with 11 & end with 11.

iii) The number of bytes that begin with 11 and not end with 11.

iv) The number of bytes begin with 11 or end with 11.

⇒ i) Since each byte contains 8 bits & each bit is 0 or 1 [two choices], the number of bytes is $2^8 = 256$

ii) In a byte beginning and ending with 11, there are 4 open positions, these can be filled in $2^4 = 16$ ways.

11 abcd 11

∴ there are 16 bytes which begin and end with 11.

iii) They occur 6 position with 11, these position can be filled with 64 ways 2^6.

Since there are 16 bytes that begin with and end with 11.

The no. of bytes that start with 11 is 64 but do not end with 11 is 16

$64 - 16 = 48$

iv) The no. of bytes that end with 11 is 64 and, the no. of bytes that begin & end with 11 is 16.

∴ The no. of bytes begin or end with 11 is

$64 + 64 - 16 = \underline{\underline{112}}$

arrangements

PERMUTATIONS: [Imp]

The number of permutations of size r of n objects is given by $P(n,r) = \frac{n!}{(n-r)!}$

PROBLEMS: [7M]

1) Find the number of permutations of [all] the letters of the word SUCCESS

⇒ $n = 7$, The given word has 7 letters, of which 3 are S, 2 are C, and 1 each U and E.

$S \to 3$
$U \to 1$
$C \to 2$
$E \to 1$

$r = 6$

∴ The required no. of permutations is,

$$P(n,r) = \frac{n!}{(n-r)!}$$

$$= \frac{7!}{(7-6)!}$$

$$= \frac{7!}{3! \times 2! \times 1! \times 1!}$$

$$= \frac{7 \times 6 \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times 1}{\cancel{3} \times \cancel{2} \times \cancel{2} \times 1 \times 1 \times 1}$$

$$= \underline{\underline{420}}$$

2) Find the number of permutations of the letters of the wo[rd]
MASSASAUGA. i) In how many of these, all 4 A's are togeth[er]

ii) How many of them begin with S?

⇒ The given word has 10 letters of which 4 are A, 3 are S, and 1 each are M, U, G.

Therefore, the required no. of permutations is,

$$\frac{10!}{4! \times 3! \times 1! \times 1! \times 1!}$$

$$\frac{10 \times 9 \times 8 \times 7 \times \cancel{6}^{3} \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1} \times \cancel{3} \times \cancel{2} \times 1 \times 1 \times 1 \times 1}$$

$$= 25200$$

i) If in a permutation, all A's are to be together, we treat all of A's as one single letter. Then, the letters to be written as, $\underbrace{(AAAA)}_{1}\underbrace{(MSSSUG)}_{6}$ {which are 7 in number}

M → 1
U → 1
G → 1
S → 3

and the number of permutations is,

$$\frac{7!}{3! \times 1! \times 1! \times 1!}$$

$$= 840$$

(ii) For permutations beginning with S, there are 9 open positions to fill, where 2 are S, 4 are A, and 1 each are M, U, G.

The no. of permutations is:

$$\frac{9!}{4! \times 1! \times 1! \times 1! \times 2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 1 \times 1 \times 1 \times 2 \times 1}$$

$$= 7560$$

3) a) How many arrangements are there for all letters in the word "SOCIOLOGICAL"?

imp

b) In how many of these arrangement

i) A and G are adjacent

ii) All the vowels are adjacent

⇒ a) The given word has 12 letters of which 3 are O's, 2 are I's, 2 are L's, 2 are C's and one each are S, G, A

Therefore, the required no. of arrangements is

$$\frac{12!}{3! \times 2! \times 2! \times 2! \times 1! \times 1! \times 1!}$$

$$= \frac{12!}{3! \times 2! \times 2! \times 2!} = \underline{9979200}$$

b) i) If in an arrangement, all A and G are to be adjacent, it is possible in 2 ways such as AG and GA.

We treat AG as one single letter

Then, the letters to be written as (AG), S, O, C, I, O, L, O, I, C, (which are 11 in number). and no. of arrangements is

$$\frac{11!}{3! \times 2! \times 2! \times 2! \times 1! \times 1!}$$

$= 831600$

In two ways, no. of arrangements $= 831600 \times 2 = 1663200$ ways

ii) If in an arrangement, all the vowels to be adjacent, we treat all of the vowels as one letter, then the letters can be written as,

(O, I, O, O, I, A), S, C, C, L, L, G and the no. of arrangements

$$\frac{7!}{1! \times 2! \times 2! \times 1! \times 1!} \times \frac{6!}{3! \times 2! \times 1!}$$

$= 42 \times 30 \times 30 \times 2$

$= 84 \times 900$

$= \underline{75600}$ ways

4) How many +ve integers n can be found using the digits 3, 4, 4, 5, 5, 6, 7. If we want n to exceed 5000000.

⇒ Here, n must be of the form

$$n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$$

where, $x_1, x_2, x_3, x_4, x_5, x_6, x_7$ are the given digits with $x_1 = 5, 6$ or 7.

then, $x_2, x_3, x_4, x_5, x_6, x_7$ is an arrangement of the remaining 6 digits which contains 2 4's, and 1 each are 3, ~~4~~, 5, 6, 7.

The number of such arrangements is,

$$\frac{6!}{1! \times 2! \times 1! \times 1! \times 1!} = 360$$

Suppose, if we take $x_1 = 6$, then $x_2 \cdots x_7$ is an arrangement of 6 digits which contains 2 each of 4 and 5 & one each of 3 and 7.

The no. of such arrangements is,

$$\frac{6!}{2! \times 2! \times 1! \times 1!} = 180$$

Similarly, suppose if we take $x_1 = 7$, then $x_2 \cdots x_7$ is an arrangement of 6 digits which contains 2 each of 4 and 5 & one each of 3 and 7.

No. of Q.P contains		No. of questions choosen		
		I	II	III
A	4	2	2	3
B	5	2	3	2
C	6	3	2	2

$^4C_2 = \frac{4!}{(4-2)!2!}$

$^4C_2 = \frac{4!}{2!\times 2!}$

No. of ways of selection,

$$\underbrace{^4C_2 \times {^5C_2} \times {^6C_3}}_{S_1} + \underbrace{^4C_2 \times {^5C_3} \times {^6C_2}}_{S_2} + \underbrace{^4C_3 \times {^5C_2} \times {^6C_2}}_{S_3}$$

$S_1 = 1200$

$S_2 = 900$

$S_3 = 600$

$\therefore$ Total number of ways $= \underline{2700}$ $[S_1 + S_2 + S_3]$

2) i) Find the number of arrangements of all the letters in TALLAHASSEE.

imp

ii) How many of these arrangements have no adjacent A's?

⇒ i) The given word has 11 letters of which 3 are A, 2 are L, 2 are S, 2 are E, one each are T and H.

Therefore, the required no. of permutations is,

$$\frac{11!}{3! \times 2! \times 2! \times 2! \times 1! \times 1!}$$

$$= \underline{\underline{831600}}$$

ii) If we discard , the A's, the remaining 8 letters can be arranged in,

T L L H S S E E

$$\frac{8!}{2! \times 2! \times 2! \times 1! \times 1!} = 5040 \text{ ways}$$

A T A L A L H S S E E

In each of these arrangements, there are 9 possible positions / locations for the 3 A's. These locations can be choosen in $^9C_3$ (or) $C(9,3)$. ⇒ 84

Therefore, the no. of arrangements having no adjacent A's is

$$5040 \times {}^9C_3 \; [C(9,3)]$$

$$= 5040 \times 84$$

$$= \underline{\underline{423360}}$$

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3) A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways, can she invite them in the following situations?

imp

i) There is no restriction on the choice?

ii) 2 particular persons will not attend separately?

iii) 2 particular persons will not attend together?

⇒ i) Since there is no restriction in the choice

5 out of 11 can be invited in $c(11,5)$

$$\frac{11!}{6! \times 5!} = 462$$

ii) Since, 2 particular persons will not attend separately they should both be invited or not invited. If both of them are $^9C_3 + {^9C_5}$ invited, then 3 more invitees are to be selected from the remaining 9 relatives. This can be done in $c(9,3)$.

$$\frac{9!}{6! \times 3!} = 84$$

If both of them are not invited, then 5 invitees are to be selected from 9 relatives. This can be done in $c(9,5)$.

$$\frac{9!}{5! \times 4!} = 126$$

Therefore, the total no. of ways in which is $84+126 \Rightarrow \underline{\underline{210}}$

iii) Since 2 particular persons P_1 and P_1 will not attend together, only one of them can be invited or none of them can be invited.

The no. of ways of choosing the invitees with P_1 (person1)

$$c(9,4) = \frac{9!}{5! \times 4!} = 126$$

Similarly, the no. of ways of choosing the invitees with P_2 invited is $c(9,5) = 126$

If both P_1 and P_2 are not invited, the no. of ways of choosing the invitees is $^9C_5 = 126$

Thus, the total no. of ways $= 126 + 126 + 126 = \underline{378}$

COMBINATIONS WITH REPETITIONS:

A combination of r objects from a set of n distinct objects with repetition is given by:

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!} = C(r+n-1, n-1)$$

PROBLEMS:

1] In how many ways can one distribute 8 identical balls into 4 distinct containers so that

i) No container is left empty?

ii) The fourth container gets an odd number of balls?

⇒ i). First we distribute one ball into each container.

- Then, we distribute the remaining 4 balls into 4 containers.
- The no. of ways =

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!(n-1)!}$$

$$C(4+4-1, 4) = C(7, 4) = \frac{7!}{3! \times 4!} = 35$$

ii) If the 4th container has to get an odd no. of balls, we have to put 1, or 3 or 5 or 7 balls into it.

Suppose, if we put one ball into 4th container, then the remaining 7 balls can be distributed into 2 containers can be done in,

$$c(3+7-1, 7) = c(9, 7) = \frac{9!}{2! \times 7!} = 36$$

$n = 3$

$r = 7$

Similarly, putting 3 balls into the 4th container & the remaining 5 balls into 3 containers. can be done in

$$c(3+5-1, 5) = c(7, 5) = \frac{7!}{2! \times 5!} = 21$$

Similarly, putting 5 balls into the 4th container & the remaining 3 balls into remaining 3 containers can be done

$$c(\cancel{8}^{3}+3-1, \cancel{8}) = c(\cancel{7}, 3) = \frac{5!}{2! \times 3!} = 10$$

Similarly, putting 7 balls into the 4th container & the remaining 1 ball into the remaining 3 containers can be done in,

$$c(3+1-1, 1) = c(3, 1) = \frac{3!}{2! \times 1!} = 3$$

Total number of ways =

$36+21+10+3$

$\Rightarrow$ 70 ways

2) In how many ways can we distribute 7 apples and 6 oranges among 4 children so that each child gets atleast one apple?

$\Rightarrow$ Suppose the first give one apple to each child. This exhausts 4 apples.

The remaining 3 apples can be distributed among the 4 children. This can be done in,

$$c(4+3-1, 3) = c(6,3) = \frac{6!}{3! \times 3!} = 20$$

Also, 6 oranges can be distributed among the 4 children in

$$c(4+6-1, 6) = c(9,6) = \frac{9!}{3! \times 6!} = 84 \text{ ways}$$

$\therefore$ By the product rule, the total no. of ways $= 20 \times 84$

$= 1680$ ways

3) A message is made up of 12 different symbols and is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces between the symbols, with atleast three spaces between each pair of consecutive symbols. In how many ways can the transmitter send such a message?

⇒ The 12 symbols can be arranged in 12 factorial ways. For each of these arrangements, there are 11 positions between the 12 symbols.

Since, there must be atleast 3 spaces between successive symbols, 33 of the 45 spaces will be filled up. The remaining 12 spaces are to be filled in 11 positions. This can be done in,

$$C(11+12-1, 12) = C(22, 12)$$

By the product rule, required no. of ways $= 12! \times C(22, 12)$

$$= \cancel{12!} \times \frac{22!}{10! \times \cancel{12!}}$$

$$= 3.09744 \times 10^{14}$$

only 2 variables

BINOMIAL AND MULTINOMIAL THEOREMS:

A basic property of $c(n, r)$ is that, it is the coefficient of $x^{n-r} y^r$ in the expansion of the expression $(x+y)^n$, where x and y are any two real numbers. In other words, the expansion of $(x+y)^n$ in powers of x and y is as follows:

$$(x+y)^n = x^n + {}^nC_1\, x^{n-1} y^1 + {}^nC_2\, x^{n-2} y^2 + \cdots + {}^nC_{n-1}\, x\, y^{n-1} + y^n$$

$$= \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r \longrightarrow (1)$$

Since, ${}^nC_r = {}^nC_{n-r}$;

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} \rightarrow (2)$$

Thus, ${}^nC_r$ is the coefficient of $x^r y^{n-r}$ in the expansion of $(x+y)^n$.

${}^nC_r$ is also denoted by $\binom{n}{r}$

$$\left.\begin{aligned} \therefore (1) &\Rightarrow (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} \\ (2) &\Rightarrow (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \end{aligned}\right\} (3)$$

Expression (3) is called Binomial theorem for a +ve integral index 'n'.

The numbers $\binom{n}{r}$ for $r = 0, 1, 2, \ldots, n$ namely $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots \binom{n}{n}$ in the above result are called the Binomial coefficients.

The generalization of Binomial theorem is known as Multinomial theorem.

Statement: For +ve integers n and k, the coefficient of $x_1^{n_1}, x_2^{n_2} \cdots x_n^{n_k}$ in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$ where each $n_i \leq n$ and

$n_1 + n_2 + \cdots + n_k = n$.

NOTE:

1) The general term in the expansion of $(x_1 + x_2 + \cdots + x_k)^n$ is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!} x_1^{n_1} x_2^{n_2} x_3^{n_3} \cdots x_k^{n_k}$$ [Alternate statement of Multinomial theorem]

2) The expression $\frac{n!}{n_1!\, n_2! \cdots n_k!}$ is also written as $\binom{n}{n_1\ n_2, \cdots n_k}$

3) Multinomial theorem can also be stated as

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{n_i} \binom{n}{n_1, n_2, \cdots n_k} x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}.$$

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FORMULAS:

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r} = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

PROBLEMS:

$\binom{n}{r} = {}^nC_r = C(n,r)$

Q] Find the coefficient of

i) x^9y^3 in the expansion of $(2x-3y)^{12}$.

ii) x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$.

$\Rightarrow$ We have, by Binomial Theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$(2x-3y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (2x)^r (-3y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^r (-3)^{12-r} x^r y^{12-r}$$

In this expansion, the co-efficient of $x^9y^3 (r=9)$ is

$$\binom{12}{9} 2^9 (-3)^3 = C(12,9) \times 2^9 \times (-3)^3$$

$$= \frac{12!}{3! \times 9!} \times 2^9 \times (-3)^3$$

$$= \frac{\cancel{12}^{2} \times 11 \times 10 \times \cancel{9!}}{\cancel{3 \times 2 \times 1} \times \cancel{9!}} \times 2^9 \times (-3)^3$$

$$= 2 \times 11 \times 10 \times 2^9 \times (-3)^3$$

$$= 11 \times 10 \times 2^{10} \times (-3)^3$$

vi) $\left(3x^2 - \frac{2}{x}\right)^{15}$

We have, by Binomial Theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} \binom{15}{r} (3x^2)^r \left(\frac{-2}{x}\right)^{15-r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} \cdot x^{2r} \cdot x^{-(+15-r)}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} \times x^{2r-15+r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r (-2)^{15-r} \times x^{3r-15}$$

In this Expansion, the coefficient of $x^0 (r=5)$

$$= \binom{15}{5} 3^5 (-2)^{15-5}$$

$$= C(15,5)\, 3^5 (-2)^{10}$$

$$= \frac{15!}{10! \times 5!} \times 3^5 \times (-2)^{10}$$

2) Find the coefficient of

i) x^9y^3 in the expansion of $(x+2y)^{12}$

ii) x^5y^2 in the expansion of $(2x-3y)^7$

iii) x^5y^2 in the expansion of $(x+y)^7$.

⇒ i) x^9y^3 in the expansion of $(x+2y)^{12}$

We have by Binomial theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Given,

$$(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} (x)^r (2y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} x^r \cdot 2^{12-r} \cdot y^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 12^{12-r} \cdot x^r \cdot y^{12-r}$$

In this expansion, the coefficient of x^9y^3 $(r=9)$ is

$$\binom{12}{9} 2^{12-9} = C(12,9) \times 2^3$$

$$= \frac{12!}{9!\,(12-9)!} \times 2^3$$

$$= \frac{12 \times 10 \times 11 \times \cancel{9!} \times 8}{\cancel{9!} \times 3!}$$

$$= \frac{\cancel{12}^{4} \times \cancel{10}^{5} \times 11}{\cancel{3} \times \cancel{2} \times 1} \times 8$$

$$= 20 \times 88$$

$$= \underline{1760}$$

ii) x^5y^2 in the expansion of $(2x-3y)^7$

$\Rightarrow$ We have by Binomial Theorem,

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Given. $(2x-3y)^7 = \sum_{r=0}^{7} \binom{7}{r} (2x)^r (-3y)^{7-r}$

$$= \sum_{r=0}^{7} \binom{7}{r} 2^r (x)^r (-3)^{7-r} \cdot y^{7-r}$$

In this expansion, the coefficient of x^5y^2 $(r=5)$ is

$$\binom{7}{5} 2^5 (-3)^{7-5} = C(7,5) \times 2^5 \times (-3)^2$$

$$= \frac{7!}{5! \times 2!} \times 2^5 \times 9$$

$$= \frac{7 \times \cancel{6}^{3} \times \cancel{5!}}{\cancel{5!} \times \cancel{2} \times 1} \times 2^5 \times 9 = \underline{\underline{6048}}$$

viii) x^5y^2 in the expansion of $(x+y)^7$.

⇒ We have by Binomial theorem.

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

Given, $(x+y)^7 = \sum_{r=0}^{7} \binom{7}{r} (x)^r (y)^{7-r}$

In this expansion, the coefficient of x^5y^2 $(r=5)$ is

$$\binom{7}{5} = C(7,5)$$

$$= \frac{7!}{5! \times 2!}$$

$$= \frac{7 \times \overset{3}{\cancel{6}} \times \cancel{5!}}{\cancel{5!} \times \cancel{2} \times 1}$$

$$= \underline{\underline{21}}$$

3) Determine the coefficient of

i) xyz^2 in the expansion of $(2x-y-z)^4$

ii) $a^2b^3c^2d^5$ in the expansion of $(a+2b+3c+2d+5)^{16}$.

$\Rightarrow$ i) By the Multinomial theorem, we have the general term

$$\binom{n}{n_1, n_2, n_3} x_1^{n_1} x_2^{n_2} x_3^{n_3}$$

Given, $n_1 = 1$, $n_2 = 1$, $n_3 = 2$, $n = 4$,

$x = 2x$

$y = -y$

$z = -z$

$$\binom{4}{1,1,2} (2x)^1 (-y)^1 (-z)^2$$

$$\binom{4}{1,1,2} \times (2)(-1)(-1)^2 \, xyz^2$$

$\therefore$ The required coefficient of xyz^2 is

$$\binom{4}{1,1,2} \times (2) \times (-1) \times (-1)^2$$

$$= \frac{4!}{1! \times 1! \times 2!} \times (2)(-1)(-1)^2$$

$$= \frac{4 \times 3 \times \cancel{2} \times 1}{1 \times 1 \times \cancel{2} \times 1} \times (-2)$$

$$= \cancel{-12} = -24$$

vi) By the Multinomial theorem, we have the general term

$$\binom{n}{n_1, n_2, n_3, n_4, n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$$

given, $a^2 b^3 c^2 d^5$ $(a+2b-3c+2d+5)^{16}$

$n = 16$, $n_1 = 2$, $n_2 = 3$, $n_3 = 2$, $n_4 = 5$, $x_1 = a$

$x_2 = 2b$

$x_3 = -3c$

$x_4 = 2d$

$x_5 = 5$

$$n_5 = n - (n_1 + n_2 + n_3 + n_4)$$

$$= 16 - 12$$

$$n_5 = 4$$

$$\binom{16}{2,3,2,5,4} a^2 (2b)^3 (-3c)^2 (2d^5)(5)^4$$

$$= \binom{16}{2,3,2,5,4} 2^3 \times (-3)^2 \times 2^5 \times 5^4 \times a^2 \times b^3 \times c^2 \times d^5$$

$\therefore$ The required coefficient of $a^2 b^3 c^2 d^5$ is

$$\binom{16}{2,3,2,5,4} 2^3 \times 3^2 \times 2^5 \times 5^4$$

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$$= \frac{16!}{2! \times 3! \times 2! \times 5! \times 4!} \times 2^8 \times 3^2 \times 5^4$$

$$= \frac{16!}{2 \times 3 \times 2 \times 1 \times 2 \times 5 \times (4!)^2} \times 2^8 \times 3^2 \times 5^4$$

$$= \frac{16!}{(4!)^2} \times 3 \times 2^5 \times 5^3$$