Subject: Discrete Mathematical Structures

Subject Code: BCS405A

Semester: 4th

Scheme: 2022

Module-4: The Principal of Inclusion & Exclusion

- Principle of Inclusion and Exclusion
- Derangements
- Rook Polynomials
- First-order Recurrence Relations
- Second-order Homogeneous Recurrence Relations with Constant Coefficients

Principles of Counting - II

Recall :- 1) $|A \cup B| = |A| + |B| - |A \cap B|$ 2) $|\overline{A}| = |S| - |A|$, S is the universal set

2) $\overline{A} \cap \overline{B} = \overline{A \cup B}$, and $|\overline{A \cup B}| = |S| - |A \cup B|$ ✓

3) $|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = |S| - |A \cup B|$

$$|\overline{A \cup B}| = |S| - |A| - |B| + |A \cap B| \rightarrow (1)$$

Eqⁿ (1) is called the addition principle (or) Principle of inclusion - exclusion for 2 sets.

Principle of Inclusion - Exclusion for 'n' sets :-

Let S be a finite set & $A_1, A_2, \ldots A_n$ be subsets of S, then

$$|\overline{A_1 \cup A_2 \cup \ldots \cup A_n}| = |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| \pm \ldots + (-1)^n |A_1 \cap A_2 \cap \ldots \cap A_n| \rightarrow (2)$$

Note :-

2) The no. of elements in S that satisfy exactly 'm' of the given 'n' condⁿˢ $(0 \le m \le n)$ is given by

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} \mp \ldots + (-1)^{n-m} \binom{n}{n-m} S_n$$

3) The no. of elements in S that satisfy atleast 'm' of the 'n' condⁿˢ $(1 \le m \le n)$ is given by

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \mp \ldots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

1) Let $S_0 = |S|$, $S_1 = \sum |A_i|$,

$S_2 = \sum |A_i \cap A_j|$, $S_3 = \sum |A_i \cap A_j \cap A_k|$

..... $S_n = |A_1 \cap A_2 \cap \ldots \cap A_n|$, then (2) becomes

$$|\overline{A_1 \cup A_2 \cup \ldots \cup A_n}| = S_0 - S_1 + S_2 \mp \ldots + (-1)^n S_n$$

Problems :-

1) Among the students in a hostel, 12 students study mathematics (A), 20 study physics (B), 20 study chemistry (C) & 8 study biology (D). There are 5 students for A & B, 7 for A & C, 4 for A & D, 16 for B & C, 4 for B & D, 3 for C & D, ~~3 for A, B & C~~. There are 3 students for A, B & C, 2 for A, B & D, 2 for B, C & D, 3 for A, C & D. finally there are 2 who study all of these subjects. further, there are 71 students who do not study any of these subjects. Find the total no of students in the hostel.

Soln :- by data, $|A| = 12$, $|B| = 20$, $|C| = 20$, $|D| = 8$

$|A \cap B| = 5$, $|A \cap C| = 7$, $|A \cap D| = 4$, $|B \cap C| = 16$, $|B \cap D| = 4$

$|C \cap D| = 3$, $|A \cap B \cap C| = 3$, $|A \cap B \cap D| = 2$, $|B \cap C \cap D| = 2$

$|A \cap C \cap D| = 3$, $|A \cap B \cap C \cap D| = 2$,

$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = 71$.

To find $|S|$, where S is the set of all students in the hostel.

we have

$$|\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D}| = |\overline{A \cup B \cup C \cup D}|$$

$$= |S| - |A \cup B \cup C \cup D|$$

$$= |S| - \left[|A| + |B| + |C| + |D|\right] + \left[|A \cap B| + |A \cap C| + |A \cap D| + |B \cap C| + |B \cap D| + |C \cap D|\right]$$

$$- \left[|A \cap B \cap C| + |A \cap B \cap D| + |B \cap C \cap D| + |A \cap C \cap D|\right] + |A \cap B \cap C \cap D|$$

$$\Rightarrow 71 = |S| - [12 + 20 + 20 + 8] + [5 + 7 + 4 + 16 + 4 + 3] - [3 + 2 + 2 + 3] + 2$$

$\Rightarrow$ $|S| = 100$ $\Rightarrow$ total no of students in the hostel is 100.

2) In the above example, among all the subjects find how many study (i) exactly one subject (ii) exactly 2 subjects (iii) exactly 3 subjects (iv) at least 1 subject (v) atleast 2 subjects (vi) atleast 3 subjects. (2)

Soln:- WKT

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} \mp \dots + (-1)^{n-m} \binom{n}{n-m} S_n \rightarrow (1)$$

is the no of students studying exactly 'm' subjects out of 'n'.

$$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \mp \dots + (-1)^{n-m} \binom{n-1}{m-1} S_n \rightarrow (2)$$

is the no of students studying atleast 'm' subjects out of 'n'

we have $S_0 = |S| = 100$. $S_1 = |A|+|B|+|C|+|D| = 60$.

$S_2 = |A \cap B| + |B \cap C| + |C \cap D| + |A \cap D| + |B \cap D| + |A \cap C| = 39$

$S_3 = |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| = 10$.

$S_4 = |A \cap B \cap C \cap D| = 2$.

(i) $E_1 = S_1 - \binom{2}{1} S_2 + \binom{3}{2} S_3 \mp \binom{4}{3} S_4 = 60 - 2 \times 39 + 3 \times 10 - 4 \times 2$

$E_1 = 4$

(ii) $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 = 39 - 3 \times 10 + 6 \times 2 = 21$

(iii) $E_3 = S_3 - \binom{4}{1} S_4 = 10 - 4 \times 2 = 2$.

(iv) $L_1 = S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 - \binom{3}{0} S_4 = 60 - 39 + 10 - 2 = 29$

(v) $L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4 = 39 - 2 \times 10 + 3 \times 2 = 25$

(vi) $L_3 = S_3 - \binom{3}{2} S_4 = 10 - 3 \times 2 = 4$.

3) DO out of 30 students in a hostel, 15 study History, 8 study Economics, 6 study Geography. It is known that 3 students study all these subjects. Show that there are 7 or more students who study none of the subjects.

P.T.O.

Soln :- $|S|$ = no of students in the hostel = 30.

Let A, B, C represent students studying history, Economics & Geography resp. then

$|A| = 15, \quad |B| = 8, \quad |C| = 6, \quad |A\cap B\cap C| = 3$

we have $|\bar{A}\cap\bar{B}\cap\bar{C}| = |S| - [|A|+|B|+|C|] + [|A\cap B| + |B\cap C| + |A\cap C|] - |A\cap B\cap C|$

$\Rightarrow |\bar{A}\cap\bar{B}\cap\bar{C}| = 30 - (15+8+6) + (|A\cap B| + |B\cap C| + |A\cap C|) - 3$

$= |A\cap B| + |B\cap C| + |A\cap C| - 2$

WKT $|A\cap B| \geq |A\cap B\cap C|$, $|B\cap C| \geq |A\cap B\cap C|$,

$|A\cap C| \geq |A\cap B\cap C|$.

$\therefore |\bar{A}\cap\bar{B}\cap\bar{C}| \geq 3|A\cap B\cap C| - 2 = 3\times 3 - 2 = 7$.

4) Determine the no of positive integers 'n', $1\leq n\leq 100$ such that

(i) n is not divisible by 2, 3 or 5.

(ii) n is divisible by atleast two of 2, 3 or 5

(iii) n is divisible by exactly two of 2, 3 or 5.

Soln :- $S = \{1, 2, 3 \ldots 100\}$ $\therefore |S| = 100$. $\therefore S_0 = 100$.

Let A — set of no's divisible by 2

B — " ——— " ——— " 3

C — " ——— " ——— " 5

(i) To find $|\bar{A}\cap\bar{B}\cap\bar{C}|$ ie $|\overline{A\cup B\cup C}|$

$\therefore |A|$ = no of elements in S that are divisible by 2

ie $|A| = \left\lfloor\frac{100}{2}\right\rfloor = 50$.

IIIly $|B| = \left\lfloor\frac{100}{3}\right\rfloor = \lfloor 33.33\rfloor = 33$

$|C| = \left\lfloor\frac{100}{5}\right\rfloor = 20$.

$\therefore S_1 = |A|+|B|+|C| = 50+33+20 = 103$

$|A \cap B|$ = no of elements in S divisible by both 2 & 3 (ie LCM = 6)

$$|A \cap B| = \left\lfloor \frac{100}{6} \right\rfloor = \lfloor 16.66 \rfloor = 16.$$

$$|A \cap C| = \left\lfloor \frac{100}{10} \right\rfloor = 10.$$

$$|B \cap C| = \left\lfloor \frac{100}{15} \right\rfloor = 6. \quad \therefore \quad S_2 = 16 + 10 + 6 = 32$$

$|A \cap B \cap C|$ = no of elements in S divisible by all 2, 3, 5 (ie LCM = 30) $= \left\lfloor \frac{100}{30} \right\rfloor = 3.$

$$\therefore S_3 = 3$$

$$\therefore |\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - (|A| + |B| + |C|) + (|A \cap B| + |B \cap C| + |A \cap C|) - |A \cap B \cap C|$$

write alternate

$$= 100 - (50 + 33 + 20) + (16 + 10 + 6) - 3$$

$$= 26.$$

$\therefore$ No of integers $1 \leq n \leq 100$, not divisible by ~~any~~ 2, 3 or 5 is 26.

(ii) $$L_m = S_m - \binom{m}{m-1} S_{m+1} + \binom{m+1}{m-1} S_{m+2} \mp \ldots + (-1)^{n-m} \binom{n-1}{m-1} S_n$$

$$\therefore L_2 = S_2 - \binom{2}{1} S_3$$

$$= |A \cap B| + |B \cap C| + |A \cap C| - 2 \times |A \cap B \cap C|$$

$$= 16 + 10 + 6 - 2 \times 3$$

$$= 26.$$

(iii) $$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} \mp \ldots + (-1)^{n-m} \binom{n}{n-m} S_n$$

$$\therefore E_2 = S_2 - \binom{3}{1} S_3 \cancel{+ \binom{4}{2}}$$

$$= |A \cap B| + |B \cap C| + |A \cap C| - 3 \times 3$$

$$= 16 + 10 + 6 - 9$$

$$= 23$$

P.T.O

5) How many integers b/w 1 & 300 (inclusive) are

HW (i) divisible by atleast one of 5, 6, 8 ?

(ii) " ——— " none of 5, 6, 8 ?

(iii) divisible by exactly two of 5, 6, 8

(iv) divisible by atleast two of 5, 6, 8 .

Soln :- $|S| = 300$.

let A, B, C be the no of integers divisible by 5, 6, 8 resp

then $|A| = \left\lfloor \frac{300}{5} \right\rfloor = 60$, $|B| = \left\lfloor \frac{300}{6} \right\rfloor = 50$, $|C| = \left\lfloor \frac{300}{8} \right\rfloor = 37$

$|A \cap B| = \left\lfloor \frac{300}{30} \right\rfloor = 10$, $|B \cap C| = \left\lfloor \frac{300}{24} \right\rfloor = 12$.

$|A \cap C| = \left\lfloor \frac{300}{40} \right\rfloor = 7$, $|A \cap B \cap C| = \left\lfloor \frac{300}{120} \right\rfloor = 2$

$\therefore$ $S_1 = |A| + |B| + |C| = 147$

$S_2 = |A \cap B| + |B \cap C| + |A \cap C| = 29$.

$S_3 = |A \cap B \cap C| = 2$.

(i) No of integers b/w 1 & 300 divisible by atleast one of 5, 6, 8 is $L_1 = S_1 - \binom{1}{0} S_2 + \binom{2}{0} S_3 = 148 - 29 + 2 = 120$

(ii) No of integers divisible by none of 5, 6, 8 is

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |S| - S_1 + S_2 - S_3 = 300 - 145 + 29 - 2 = 180$

(iii) No of integers divisible by exactly two of 5, 6, 8 is

$E_2 = S_2 - \binom{3}{1} S_3 = 29 - 3 \times 2 = 23$

(iv) No of integers divisible by atleast two of 5, 6, 8 is

$L_2 = S_2 - \binom{2}{1} S_3 = 29 - 2 \times 2 = 25$.

6) In how many ways 5 no of a's, 4 no of b's & 3 no of c's can be arranged so that all identical letters are not in a single block?

DO

Soln:- Altogether there are 12 letters, in which 5 are a's, 4 are b's, 3 are c's.

Let S be the set of all arrangements of the 12 letters.

$\therefore \quad |S| = \dfrac{12!}{5!\,4!\,3!} = 27{,}720$.

Let A_1 be the set of all arrangements in which all 5 a's are in a single block.

$\underbrace{\boxed{aaaaa}}_{1X}$ bbbb ccc

ie $|A_1| = \dfrac{8!}{4!\,3!\,1!} = 280$.

|||ly $|A_2|$ = no of ways of arranging in which all 4 b's are together. ie $|A_2| = \dfrac{9!}{5!\,1!\,3!} = 504$

$|A_3|$ = no of ways of arranging in which all 3 c's are together ie $|A_3| = \dfrac{10!}{5!\,4!\,1!} = 1260$.

Also $|A_1 \cap A_2|$ = no of ways of arranging in which all 5 a's & all 4 b's are together.

$\underbrace{aaaaa}_{1X}$ $\underbrace{bbbb}_{1Y}$ ccc

ie $|A_1 \cap A_2| = \dfrac{5!}{1!\,1!\,3!} = 20$

aaaaa $\underbrace{bbbb}_{1X}$ $\underbrace{ccc}_{1Y}$

|||ly $|A_2 \cap A_3| = \dfrac{7!}{5!\,1!\,1!} = 42$

$\underbrace{aaaaa}_{1X}$ bbbb $\underbrace{ccc}_{1Y}$

$|A_1 \cap A_3| = \dfrac{6!}{1!\,4!\,1!} = 30$

$\underbrace{aaaaa}_{1X}$ $\underbrace{bbbb}_{1Y}$ $\underbrace{ccc}_{1Z}$

and $|A_1 \cap A_2 \cap A_3| = 3! = 6$.

$\therefore \quad |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}| = |S| - \left[|A_1| + |A_2| + |A_3|\right] + \left[|A_1 \cap A_2| + |A_2 \cap A_3| + |A_1 \cap A_3|\right] - |A_1 \cap A_2 \cap A_3|$

$= 27{,}720 - \left[280 + 504 + 1260\right] + \left[20 + 42 + 30\right] - 6$

$= 25{,}762$

Do 6th, then 7th.

7) Find the no of permutations of letters a, b, c ... x, y, z in which none of the patterns spin, game, path or net occurs.

DO

Soln:- $|S| = 26!$

$|A_1|$ = no of permutations containing the pattern 'spin'

$= \cancel{(26-4+1)}! = 23!$

SPIN 1X ~~22 26-4 letters in spin = 22~~ ~~∴ 22+1X = 23~~

$|A_2|$ = no of permutations containing the pattern 'game'.

$= \cancel{(26-4+1)}! = 23!$

$|||^{ly}$ $|A_3| = 23!$

$|A_4| = \cancel{(26-3+1)}! = 24!$

SPIN GAME - all distinct
$\underbrace{SPIN}_{1X}$ $\underbrace{GAME}_{1Y}$ 18
~~((26-8)+2)! = 20!~~

$\therefore |A_1 \cap A_2| = \cancel{((26-8)+2)}! = 20!$

$|A_1 \cap A_3| = 0$ ($\because$ no pattern can contain spin & path) — p repeated & cannot be made a pattern since in word spin, p is not at end.

$|A_1 \cap A_4| = ((26-6)+1)! = 21!$ → $\underbrace{SPINET}_{1X}$ 20.

$|A_2 \cap A_3| = 0$, $|A_2 \cap A_4| = 0$, $|A_3 \cap A_4| = 0$.

$|A_1 \cap A_2 \cap A_3| = 0$, $|A_2 \cap A_3 \cap A_4| = 0$, $|A_1 \cap A_3 \cap A_4| = 0$

$|A_1 \cap A_2 \cap A_4| = 0$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0.$

$\therefore$ Req no $= |\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}|$

$= |S| - [\Sigma |A_i|] + [\Sigma |A_i \cap A_j|] - [\Sigma |A_i \cap A_j \cap A_k|]$

$+ \Sigma |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= 26! - (24! + (3 \times 23!)) + (20! + 21!)$

$= 4.026 \times 10^{26}$

8) Find the no of permutations of the digits 1 through 9 in which the blocks a) 23, 57, 468 do not appear.

Hw

b) 36, 78, 672 do not appear.

DO

Sol: a) $|S|$ = no of permutations of the digits 1 to 9

$|S| = 9!$

$|A_1|$ = no of permutations of 1 to 9 such that 23 is (present) contained in it

$= \cancel{((9-2)+1)!} = 8!$

$\underbrace{23}_{1x}$ — 7   1+7=8

$|A_2| = \cancel{((9-2)+1)!} = 8!$

$|A_3| = \cancel{(9-3+1)!} = 7!$

$\underbrace{23}_{1x}, \underbrace{57}_{1y}$

$|A_1 \cap A_2| = \cancel{(9-4+2)!} = 7!$

$|A_1 \cap A_3| = \cancel{(9-5+2)!} = 6!$

$|A_2 \cap A_3| = \cancel{(9-5+2)!} = 6!$

$\underbrace{23}_{1x}, \underbrace{57}_{1y}, \underbrace{468}_{1z}$

$|A_1 \cap A_2 \cap A_3| = \cancel{(9-7+3)} = 5!$

$\therefore$ Req No $= |\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}|$

$= |S| - [\Sigma |A_i|] + [\Sigma |A_i \cap A_j|] - \Sigma |A_i \cap A_j \cap A_k|$

$= 9! - (8! + 8! + 7!) + (7! + 6! + 6!) - 5!$

$= 2,83,560$

b) $|S| = 9!$

$|A_1|$ = no of permutations such that 36 is present in that

$= \cancel{(9-2+1)!} = 8!$

$\underbrace{36}_{1x}$ — 7

$|A_2| = \cancel{(9-2+1)!} = 8!$

$|A_3| = \cancel{(9-3+1)!} = 7!$

$\underbrace{36}_{1x}, \underbrace{78}_{1y}$ — 5

$\underbrace{36, 672}$

$\underbrace{3672}_{1x}$ — 5

78, 672

$|A_1 \cap A_2| = \cancel{(9-4+2)!} = 7!$

$|A_1 \cap A_3| = \cancel{(9-4+1)!} = 6!$

$|A_2 \cap A_3| = 0.$

and $|A_1 \cap A_2 \cap A_3| = 0.$

Thus Req no $= |\bar{A_1} \cap \bar{A_2} \cap \bar{A_3}|$

$= 9! - (2 \times 8! + 7!) + (7! + 6!)$

$= 2,82,960$

9) In how many ways can 26 letters of English alphabet be permuted so that (i) none of the patterns CAR, DOG, PUN or BYTE occurs (ii) a) exactly 2 b) exactly 3 & c) atleast 3 of patters CAR, DOG, PUN & BYTE occurs.

Soln:- $|S| = 26!$

Let A_1, A_2, A_3, A_4 be sets of permutations containing the patterns CAR, DOG, PUN, BYTE alone resp.

$\underbrace{CAR}_{1X}$ ___ 23 $23+1=24$

$|A_1| = \cancel{(26-3+1)!} = 24!$

$|A_2| = 24!$

$|A_3| = 24!$

$|A_4| = \cancel{(26-4+1)!} = 23!$

$\underbrace{CAR}_{1X}, \underbrace{DOG}_{1Y}$ ___ 20

$\underbrace{CAR}_{1X}$ $\underbrace{BYTE}_{1Y}$ ___ 19

$|A_1 \cap A_2| = \cancel{(26-6+2)!} = 22!$

$|A_1 \cap A_3| = \cancel{(26-6+2)!} = 22!$

$|A_1 \cap A_4| = \cancel{(26-7+2)!} = 21!$

$|A_2 \cap A_3| = \cancel{(26-6+2)!} = 22!$

$|A_2 \cap A_4| = \cancel{(26-7+2)!} = 21!$

$|A_3 \cap A_4| = \cancel{(26-7+2)!} = 21!$

$\underbrace{CAR}_{1X}, \underbrace{DOG}_{1Y}, \underbrace{PUN}_{1Z}$ ___ 17

$|A_1 \cap A_2 \cap A_3| = \cancel{(26-9+3)!} = 20!$

$|A_1 \cap A_2 \cap A_4| = \cancel{(26-10+3)!} = 19!$

$|A_1 \cap A_3 \cap A_4| = \cancel{(26-10+3)!} = 19!$

$|A_2 \cap A_3 \cap A_4| = \cancel{(26-10+3)!} = 19!$

& $|A_1 \cap A_2 \cap A_3 \cap A_4| = (26-13+4)! = 17!$

(i) Req no $= |\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = |\bar{A_1} \cap \bar{A_2} \cap \bar{A_3} \cap \bar{A_4}|$

$= 26! - (3\times 24! + 23!) + (3\times 22! + 3\times 21!) - (20! + 3\times 19!) + 17!$

$= 4.014 \times 10^{26}$

(ii) $E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} \mp \cdots$

$E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4$

$= (3\times 22! + 3\times 21!) - 3\times(20! + 3\times 19!) + 6\times 17!$

$= 3.517 \times 10^{21}$

(iii) $E_3 = s_3 - \binom{4}{1} s_4 = (20! + 3\times 19!) - 4 \times 17!$ (6)

$= 2.796 \times 10^{18}$

(iv) $L_m = s_m - \binom{m}{m-1} s_{m+1} + \binom{m+1}{m-1} s_{m+2} \mp \cdots$

$L_3 = s_3 - \binom{3}{2} s_4 = 3\times 19! + 20! - 3\times 17!$

$= 2.797 \times 10^{18}$

10) In how many ways one can arrange the letters in the word CORRESPONDENTS so that

(i) there is no pair of consecutive identical letters

(ii) there are exactly 2 pairs of consecutive identical letters

(iii) " ——— atleast 3 " ——— " ——— "

Soln:- $|S| = \dfrac{14!}{2!2!2!2!2!}$ ∵ given word has 1 C, 2 O's, 2 R's, 2 E's, 2 S's, 1 P, 2 N's, 1 D, 1 T

Let A1, A2, A3, A4, A5 be the sets of Permutations ~~of given word~~ in which contains 2 consecutive O's, R's, E's, S's, N's ~~appear in pairs~~ resp. Then

$|A_1| = \dfrac{13!}{(2!)^4}$ ~~(14-1=13)~~ $= |A_2| = |A_3| = |A_4| = |A_5|$.

OO 1X, 12 — 2 R's

$|A_i \cap A_j| = \dfrac{12!}{(2!)^3}$ ~~(14-2=12)~~

OO 1X, RR 1Y, 10

1C, 2 E's, 2 N's

$|A_i \cap A_j \cap A_k| = \dfrac{11!}{(2!)^2}$ ~~(14-3=11)~~ ⟶ 8, OO, RR, EE — 1X 1Y 1Z

2N's

$|A_i \cap A_j \cap A_k \cap A_p| = \dfrac{10!}{(2!)^1}$ ~~(14-4=10)~~ ⟶ 6, OO, RR, EE, SS — 1X 1Z 1Y 1

$|A_i \cap A_j \cap A_k \cap A_p \cap A_q| = \dfrac{9!}{(1!)} = 9!$. ⟶ 4, OO, RR, EE, SS, NN — 1 1 1 1 1

∴ $s_0 = |S| = \dfrac{14!}{(2!)^5}$,

$s_1 = {}^5C_1 \times \dfrac{13!}{(2!)^4}$, $s_2 = {}^5C_2 \times \dfrac{12!}{(2!)^3}$,

$s_3 = {}^5C_3 \times \dfrac{11!}{(2!)^2}$, $s_4 = {}^5C_4 \times \dfrac{10!}{(2!)^1}$, $s_5 = {}^5C_5 \times 9!$

(i) $|\overline{A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5}| = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$

$= \overset{S_0}{|S|} - S_1 + S_2 - S_3 + S_4 - S_5$

$= \frac{14!}{(2!)^5} - \frac{5 \times 13!}{(2!)^4} + \frac{10 \times 12!}{(2!)^3} - \frac{10 \times 11!}{(2!)^2} + \frac{5 \times 10!}{2} - 9!$

$= 1,28,60,46,720$

(ii) $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 - \binom{5}{3} S_5$

$= 10 \times \frac{12!}{(2!)^3} - 3 \times 10 \times \frac{11!}{(2!)^2} + 6 \times 5 \times \frac{10!}{2!} - 10 \times 9!$

$= 35,01,79,200$

(iii) $L_2 = S_3 - \binom{3}{2} S_4 + \binom{4}{2} S_5$

$= \frac{10 \times 11!}{(2!)^2} - 3 \times 5 \times \frac{10!}{2!} + 6 \times 9!$

$= 7,47,53,280$

11) In how many ways can the integers 1, 2, 3, ... 10 be arranged in a line so that no even integer is in its natural place.

Soln:- $S = \{1, 2, \ldots 10\} \Rightarrow |S| = 10!$ (∵ permutations)

Let A_1 be the set of all permutations of the given integers, where 2 is in its natural place.

A_2 be the set of all permutations in which 4 is in its natural place

& so on.

To find $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4} \cap \overline{A_5}|$

The permutations in A_1 are all of the form $b_1 2 b_3 b_4 \ldots b_{10}$ where $b_1 b_3 b_4 \ldots b_{10}$ is a permutation of $1, 3, 4, 5 \ldots 10$

∴ $|A_1| = 9!$ ∵ 2 — 9

$|||^{ly}$ $|A_2| = |A_3| = |A_4| = |A_5| = 9!$

$\therefore \quad s_1 = \Sigma |A_i| = 5 \times 9!$ (or) $5c_1 \times 9!$

The permutation in $A_1 \cap A_2$ are all of the form $b_1\, 2\, b_3\, 4\, b_5 b_6 \ldots b_{10}$, where $b_1 b_3 b_5 b_6 \ldots b_{10}$ is a permutation of $1, 3, 5, 6 \ldots 10$.

$\therefore \quad |A_1 \cap A_2| = 8!$ ② ④ $\frac{\quad}{8}$

$|||^{ly} \quad |A_i \cap A_j| = 8! \quad \forall\, i, j.$

$\therefore \quad S_2 = \Sigma |A_i \cap A_j| = 5c_2 \times 8!$

Hence $S_3 = 5c_3 \times 7!$, $S_4 = 5c_4 \times 6!$, $S_5 = 5c_5 \times 5!$

$\therefore$ Req no $= |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4 \cap \bar{A}_5|$

$= S_0 - S_1 + S_2 - S_3 + S_4 - S_5$

$= 10! - 5 \times 9! + 5c_2 \times 8! - 5c_3 \times 7! + 5c_4 \times 6! - 5c_5 \times 5!$

$= 21,70,680$

12) Find the no of non-negative integer solutions of the Eqn $x_1 + x_2 + x_3 + x_4 = 18$ under the condn $x_i \leq 7$ for $i = 1, 2, 3, 4$.

Soln :- Let S be the set of non-negative integer solutions of the given Eqn. The no of such solutions is $C(4+18-1, 18) = |S| = C(21, 18)$. $\Rightarrow$ $\boxed{S_0 = 1330}$

Let A_1 be the subset of S, that contains non-negative integer solns of the Eqn under the condn $x_1 > 7$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

$A_1 = \{ (x_1, x_2, x_3, x_4) \in S \mid \underset{x_1 \geq 8}{x_1 > 7} \}$

$|||^{ly} \quad A_2 = \{ (x_1, x_2, x_3, x_4) \in S \mid \underset{x_2 \geq 8}{x_2 > 7} \}$

$A_3 = \{ (x_1, x_2, x_3, x_4) \in S \mid \underset{x_3 \geq 8}{x_3 > 7} \}$

$A_4 = \{ (x_1, x_2, x_3, x_4) \in S \mid \underset{x_4 \geq 8}{x_4 > 7} \}$

To find $|\bar{A_1} \cup \bar{A_2} \cup \bar{A_3} \cup \bar{A_4}|$:-

let $y_1 = x_1 - 8$, then $x_1 > 7$ (or) $x_1 \geq 8 \Rightarrow y_1 \geq 0$.

$\therefore$ $x_1 + x_2 + x_3 + x_4 = 18$ becomes

$y_1 + 8 + x_2 + x_3 + x_4 = 18$

$\Rightarrow$ $y_1 + x_2 + x_3 + x_4 = 10$.

$\therefore$ No of non-negative integer solns of this Eqn is

$C(4+10-1, 10) = C(13, 10) = |A_1|$.

|||ly $|A_2| = |A_3| = |A_4| = C(13,10)$.

$\therefore$ $S_1 = {}^4C_1 \times |A_i| = {}^4C_1 \times C(13,10) \Rightarrow \boxed{S_1 = 1144}$

let $y_1 = x_1 - 8$, $y_2 = x_2 - 8$, then

$x_1 > 7 \Rightarrow y_1 \geq 0$ & $x_2 > 7 \Rightarrow y_2 \geq 0$.

$\therefore$ $x_1 + x_2 + x_3 + x_4 = 18$ becomes

$y_1 + 8 + y_2 + 8 + x_3 + x_4 = 18$ (or) $y_1 + y_2 + x_3 + x_4 = 2$

$\therefore$ No of non-negative integer solns of this Eqn $= C(4+2-1, 2)$

$= C(5,2) = |A_1 \cap A_2| = |A_2 \cap A_3| = |A_2 \cap A_4| = |A_1 \cap A_3| =$

$= |A_1 \cap A_4| = |A_3 \cap A_4|$

$\therefore$ $S_2 = {}^4C_2 \times |A_i \cap A_j| = 6 \times C(5,2) \Rightarrow \boxed{S_2 = 60}$

$|A_1 \cap A_2 \cap A_3| =$ no of non-negative integer soln of the Eqn with $x_1 > 7$, $x_2 > 7$, $x_3 > 7$. But it is not possible since the sum is 18.

$\therefore$ $|A_1 \cap A_2 \cap A_3| = 0 = |A_1 \cap A_2 \cap A_4| = |A_1 \cap A_3 \cap A_4| = 0$

$|A_2 \cap A_3 \cap A_4| = 0 \Rightarrow \boxed{S_3 = 0}$

Also $|A_1 \cap A_2 \cap A_3 \cap A_4| = 0. \Rightarrow \boxed{S_4 = 0}$

$\therefore$ Req no $= |\overline{A_1 \cup A_2 \cup A_3 \cup A_4}| = S_0 - S_1 + S_2 - S_3 + S_4$

$= |S| - \sum |A_i| + \sum |A_i \cap A_j| - \sum |A_i \cap A_j \cap A_k| + |A_1 \cap A_2 \cap A_3 \cap A_4|$

$= {}^{21}C_{18} - {}^4C_1 \times {}^{13}C_{10} + {}^4C_2 \times {}^5C_2$

$= \underline{246}$

13) Find the no of integer solutions of the Eqn $x_1+x_2+x_3=20$ Such that $2 \le x_1 \le 5$, $4 \le x_2 \le 7$, $-2 \le x_3 \le 9$.

8 (a)

Soln :- Let $y_1 = x_1 - 2$, $y_2 = x_2 - 4$, $y_3 = x_3 + 2$.

then $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$.

$\therefore$ Given Eqn becomes $y_1 + 2 + y_2 + 4 + y_3 - 2 = 20$

$$y_1 + y_2 + y_3 = 16.$$

Let $|S|$ = no of non-negative integer solutions of the Eqn $y_1+y_2+y_3=16$. No of such solns $= C(3+16-1, 16) = C(18,16)$.

ie $\boxed{|S| = C(18,16)}$

Let $A_1 = \{(y_1, y_2, y_3) \mid y_1 > 3\}$ $y_1 \geq 4$

$A_2 = \{(y_1, y_2, y_3) \mid y_2 > 3\}$ $y_2 \geq 4$

$A_3 = \{(y_1, y_2, y_3) \mid y_3 > 11\}$ $y_3 \geq 12$

[$\because$ when $x_1 \le 5$, $y_1 \le 3$.
lly when $x_2 \le 7$, $y_2 \le 3$
when $x_3 \le 9$, $y_3 \le 11$]

Let $z_1 = y_1 - 4$ then $z_1 \geq 0$ for $y_1 > 3$ (or $y_1 \geq 4$)

$\therefore$ $y_1 + y_2 + y_3 = 16$ becomes

$z_1 + 4 + y_2 + y_3 = 16 \Rightarrow z_1 + y_2 + y_3 = 12$

$\therefore$ no of non-negative integer solns of $z_1 + y_2 + y_3 = 12$ is $C(3+12-1, 12)$

$= C(14,12)$.

$\therefore$ $\boxed{|A_1| = C(14,12) = |A_2|}$.

Let $z_3 = y_3 - 12$, then $z_3 \geq 0$ for $y_3 > 11$.

$\therefore$ $y_1 + y_2 + y_3 = 16$ becomes $y_1 + y_2 + z_3 + 12 = 16$

$$y_1 + y_2 + z_3 = 4$$

non negative integer solns of $y_1 + y_2 + z_3 = 4$ is

$C(3+4-1, 4) = C(6,4)$.

$\therefore$ $\boxed{|A_3| = C(6,4)}$

$|A_1 \cap A_2| =$ no of non negative integer solns of the Eqn

$$z_1 + 4 + z_2 + 4 + y_3 = 16$$
$$z_1 + z_2 + y_3 = 8$$

$$= C(3+8-1, 8) = \underline{C(10,8)}$$

$|A_1 \cap A_3| =$ no of non-ve int solns of $z_1 + 4 + z_2 + z_3 + 12 = 16$

i.e $z_1 + z_2 + z_3 = 0$

$$= C(3+0-1, 0) = \underline{C(2,0)}$$

$|A_2 \cap A_3| =$ no of non-ve int Solns of $y_1 + z_2 + 4 + z_3 + 12 = 16$

i.e $y_1 + z_2 + z_3 = 0$

$$= \underline{C(2,0)}.$$

$|A_1 \cap A_2 \cap A_3| =$ no of non-ve int soln of the Eqn

$y_1 + y_2 + y_3 = 16$ with $y_1 > 3$, $y_2 > 3$, $y_3 > 11$

But it is not possible, since the sum is 16.

$\therefore$ $\boxed{|A_1 \cap A_2 \cap A_3| = 0}$

$\therefore$ Required no $= |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$

$$= |S| - \Sigma |A_i| + \Sigma |A_i \cap A_j| - \Sigma |A_i \cap A_j \cap A_k|$$

$$= {}^{18}C_{16} - \left[2 \times {}^{14}C_{12} + {}^{6}C_4\right] + \left[{}^{10}C_8 + 2 \times {}^{2}C_0\right]$$

$$= 3.$$

8) (b)

* In how many ways can one distribute 10 distinct prizes among 4 students with (a) exactly 2 students getting nothing (b) atleast 2 students getting nothing.

Soln :- we have to find E_2 and L_2.

we have $E_2 = S_2 - \binom{3}{1} S_3 + \binom{4}{2} S_4 \mp \cdots \longrightarrow (1)$

$$L_2 = S_2 - \binom{2}{1} S_3 + \binom{3}{1} S_4 \mp \cdots \longrightarrow (2)$$

Let A_1, A_2, A_3, A_4 be 4 students among which 10 distinct prizes are to be distributed.

$\therefore S_2 = {}^4C_2 \times |A_i \cap A_j|$

where $|A_i \cap A_j|$ = no of ways of distributing 10 distinct prizes among 4 students so that 2 students get nothing

= no of ways of distributing 10 distinct prizes among 2 stud.

$= 2^{10}$.

Prize 1 – 2 studs $\Rightarrow$ 2 ways
Prize 2 – " – " $\Rightarrow$ 2 ways
⋮
Prize 10 – 2 studs $\Rightarrow$ 2 ways
$\therefore$ Total $= 2 \times 2 \times \cdots \times 2$ (10 times)
$= 2^{10}$

$\therefore \boxed{S_2 = {}^4C_2 \times 2^{10} = 6144}$.

Also $S_3 = {}^4C_3 \times |A_i \cap A_j \cap A_k|$

where $|A_i \cap A_j \cap A_k|$ = no of ways of distributing 10 distinct prizes among 4 students, so that 3 students get nothing

= no of ways of distributing 10 distinct prizes among 1 stud

$= 1^{10}$

$\therefore \boxed{S_3 = {}^4C_3 \times 1^{10} = 4}$

Note that $S_4, S_5 \ldots$ are all zero's.

(1) & (2) $\Rightarrow$ $E_2 = 6144 - 3 \times 4 \Rightarrow \boxed{E_2 = 6132}$

$L_2 = 6144 - 2 \times 4 \Rightarrow \boxed{L_2 = 6136}$

Derangements :- A permutation of 'n' objects in which none of the objects is in its natural place (original place) is called a derangement.

Ex:- Permutation of 'n' distinct objects $1, 2, 3 \ldots n$ in which 1 is not in the 1^{st} place, 2 is not in the 2^{nd} place, ... n is not in the n^{th} place is a derangement.

The no of possible derangements of 'n' distinct objects ~~1, 2, 3 ... n~~ is denoted by d_n and is given by the formula

~~Note :- 1) $d_1 = 0$, $d_2 = 1$, $d_3 = 2$~~

$$d_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\Rightarrow d_n = n! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \pm \cdots \frac{(-1)^n}{n!}\right] \rightarrow (1)$$

$\Rightarrow d_1 = 0, \quad d_2 = 1, \quad d_3 = 2$ etc.

Note :-

1) we have $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$$\therefore e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \pm \cdots \approx 0.3679.$$

$$\therefore d_n = n! \times e^{-1} \quad \text{for } n \geq 7.$$

($\because$ for $n \geq 7$, $1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{7!} \doteqdot 0.36786$)

Problems :-

1) find the no of derangements of 1, 2, 3, 4. List all derangements

Soln :-

$$d_4 = 4! \times \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right]$$

$$= 4! \times \left[\cancel{1} - \cancel{1} + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right]$$

$$= 24 \times \frac{(12 - 4 + 1)}{24}$$

$$\begin{array}{r|l} 2 & 2, 6, 24 \\ \hline 3 & 1, 3, 12 \\ \hline & 1, 1, 4 \end{array}$$

$\boxed{d_4 = 9}$ The derangements are:

2 3 4 1	3 4 1 2	4 1 2 3
2 4 1 3	3 4 2 1	4 3 1 2
2 1 4 3	3 1 4 2	4 3 2 1

2) How many derangements are there for 1, 2, 3, 4, 5

Soln:- $d_5 = 5! \times \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right\}$

$= 120 \times \left[\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right]$

$= 120 \times \frac{(60 - 20 + 5 - 1)}{120}$

$= 44.$

3) In how many ways one can arrange the no's 1, 2, 3 ... 10 so that 1 is not in 1st place, 2 is not in 2nd place ... 10 is not in 10th place?

Soln :- $d_{10} = 10! \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{10!}\right\}$

$= 10! \times e^{-1} \simeq 13,35,036.$

$\Big(\because \quad e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$

$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} \mp \dots \cong 0.3679\Big).$

4) List all derangements of no's 1, 2, 3, 4, 5, 6 where first 3 no's are 1, 2, 3 in some order.

Soln :- The no of derangements of 1, 2, 3 in some order is d_3.

" ———————— " of 4, 5, 6 is d_3.

$\therefore$ Total no of derangements of given no's where first 3 no's are 1, 2, 3 in some order is $d_3 \times d_3 = d_3^2 = 2^2 = 4$

List of such derangements are:

2	3	1	5	6	4
3	1	2	6	4	5
2	3	1	6	4	5
3	1	2	5	6	4

5) For the integers 1, 2, 3 . . . n, there are 11660 derangements where 1, 2, 3, 4, 5 appear in the first 5 positions. What is the value of n?

Soln :- The integers 1, 2, 3, 4, 5 can be deranged in the first 5 places in d_5 ways. The remaining n−5 integers in d_{n-5} ways.

∴ No of derangements $= d_5 \times d_{n-5} = 11660$ (given)

$$\Rightarrow d_{n-5} = \frac{11660}{d_5} = \frac{11660}{44} = 265.$$

$$\Rightarrow (n-5)! \times e^{-1} = 265.$$

$$\Rightarrow (n-5)! = e \times 265$$

$$\Rightarrow (n-5)! = 720 = 6!$$

$$\Rightarrow n-5 = 6 \Rightarrow \boxed{n = 11}$$

6) There are 8 letters to 8 distinct people to be placed in 8 different addressed envelopes. Find the no of ways of doing this so that atleast one letter goes (reaches) to the right person.

Soln :- The No of ways of placing 8 letters in 8 envelopes is 8!

The no of ways of placing 8 letters in 8 envelopes such that no letter is in the right envelope is d_8.

∴ No of ways of placing 8 letters in 8 envelopes such that atleast one letter reaches the right person ~~is in the right envelope~~ is

$$8! - d_8 = 8! - [8! \times e^{-1}]$$

$$= 8! [1 - e^{-1}] = 8! [1 - 0.3679]$$

$$= 25486.$$

7) Each of the n students is given a book. The books are to be returned & redistributed to the same students. In how many ways can the 2 distributions be made so that no student will get the same book in both the distributions.

Soln:- first time, distribution can be made in $n!$ ways.

No of ways of redistribution so that no students gets the same book $= d_n$.

$\therefore$ No of ways of 2 distributions such that no student gets the same book in both distribution

$$= n! \times d_n$$

$$= n! \times n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= (n!)^2 \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

8) from the set of all permutations of n distinct objects, one permutation is chosen at random. what is the probability that it is not a derangement?

Soln:- No of permutations of n distinct objects is $n!$ (possible outcomes)

No of derangements of these objects is d_n (favourable outcomes)

$\therefore$ Prob that a permutation chosen is not a derangement is

$$p = 1 - \frac{d_n}{n!} = 1 - \left\{1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \pm \dots + \frac{(-1)^n}{n!}\right\}$$

$$p = 1 - \frac{1}{2!} + \frac{1}{3!} \mp \dots - \frac{(-1)^n}{n!}$$

P.T.O.

(9) There are 'n' pairs of children's gloves in a box. Each pair is of a different color. Suppose the right gloves are distributed at random to 'n' children, and thereafter left gloves are also distributed to them in random. find the probability that: (11)

(i) no child gets a matching pair

(ii) every child gets a matching pair

(iii) Exactly one child gets a matching pair.

(iv) atleast 2 children get matching pairs.

Soln:- The left gloves can be distributed to n children in $n!$ ways.

(i) The event of no child getting a matching pair occurs if the distribution of the left gloves is a derangement, the no of derangements $= d_n$.

$\therefore$ req prob is $p_1 = \frac{d_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} \mp \dots + \frac{(-1)^n}{n!}$.

(ii) The event of every child getting a matching pair occurs in only one distribution of left gloves.

$\therefore$ Required Prob $= p_2 = \frac{1}{n!}$

(iii) The event of exactly one child getting a matching pair occurs when only one left glove is in the natural place & all others in wrong places. The no of such distributions is d_{n-1}.

$\therefore$ Req prob $= p_3 = \frac{d_{n-1}}{n!} = \frac{1}{n!}\left[(n-1)!\left\{1 - \frac{1}{1!} + \frac{1}{2!} \mp \dots + \frac{(-1)^{n-1}}{(n-1)!}\right\}\right]$

$p_3 = \frac{1}{n}\left[1 - \frac{1}{1!} + \frac{1}{2!} \mp \dots + \frac{(-1)^{n-1}}{(n-1)!}\right]$ $\quad [\because n! = n(n-1)!]$

(iv) The event of atleast 2 children getting a matching pair occurs if the event of no child (or) one child getting a matching pair does not occur.

$\therefore$ Req Prob $= p_4 = 1 - (p_1 + p_3)$.

10) Find the no. of derangements of the integers from 1 to 2n satisfying the condⁿ that the elements in first 'n' places are

i) 1, 2, 3 ... n in some order.

ii) n+1, n+2, ... 2n in some order.

Solⁿ:- i) The no. of derangements of 1, 2, 3 ... n in first 'n' places in some order is d_n.

and " ———— " of (n+1), (n+2) ... 2n is — d_n

$\therefore$ Total no. of derangements $= d_n \times d_n = d_n^2$.

ii) The no. of derangements of (n+1), (n+2) ... 2n in first 'n' places in some order is $n!$.

and no. of derangements of 1, 2 ... n is $n!$.

$\therefore$ Total no. of derangements $= n! \times n!$
$= (n!)^2$.

(Ex:- 1, 2, 3, 4, 5, 6
Que:- 4, 5, 6 in first 3 places
then every arrangement of 4, 5, 6 in first 3 places is a derangement so this can be done in 3! ways.)

Rook (or) Pawn (or) Castle

Rook Polynomials :-

- Consider a board that represents ~~a full chess board or~~ a part of a chess board.
- Let 'n' be the no of squares present in the board.
- Pawns (Rooks) are placed on the board such that not ~~pawns~~ more than one pawn occupies a square. Then we cannot use more than 'n' pawns.
- Two pawns (Rooks) placed on a board are said to capture (or take) eachother if they are in the same row or same column of the board.
- Let r_k denote the no of ways in which 'k' rooks ~~pawns~~ can be placed on a board such that no two rooks ~~pawns~~ capture eachother
- If the board is denoted by 'C', then the rook polynomial is given by $r(C,x) = 1 + nx + r_2x^2 + \dots + r_nx^n$.

Always $r_1 = n$ where n is the no. of square in given board. ~~(Here n = r)~~

[$\because$ r_1 denotes the no of ways 1 Pawn can be placed on a board 1 Pawn can be placed anywhere on the board. $\therefore$ $r_1 = n$]

- Rook polynomial is defined for $n \geq 2$. If $n=1$, then the board contains only 1 square $\therefore$ $r_2, r_3 \dots$ are zeroes.

$\therefore$ $r(C,x) = 1 + x$.

- Rook polynomial for $n \times n$ board is given by:

$$r(C_{n\times n}, x) = 1 + \binom{n}{1}^2 x + 2! \times \binom{n}{2}^2 x^2 + 3! \times \binom{n}{3}^2 x^3 + \dots + n! \times \binom{n}{n}^2 x^n.$$

Expansion formula :-

In a given board 'C', suppose we choose a particular square & mark it as ⊛. Let 'D' be the board obtained from 'C' by deleting the row & column containing the square ⊛ and 'E' be the board obtained from C by deleting only the square ⊛, then the rook polynomial for the board 'C' is given by

$$r(C,x) = x\, r(D,x) + r(E,x).$$

This is known as Expansion formula for $r(C,x)$.

√ Product formula :- Suppose a board is made up of 2 parts C_1 & C_2 where C_1 & C_2 have no square in the same row or column of C. Such parts are called disjoint subboards of C. Then the rook polynomial for the board C is given by

$$r(C,x) = r(C_1,x) \times r(C_2,x).$$

This is known as product formula for $r(C,x)$.

X(If a board is made up of Pairwise disjoint subboards $C_1, C_2, \ldots C_n$ then $r(C,x) = r(C_1,x) \times r(C_2,x) \times \ldots \times r(C_n,x)$)X

Arrangements with forbidden positions :-

Suppose 'm' objects are to be arranged in 'n' places where $n \geq m$. Suppose there are constraints under which some objects cannot occupy certain places. Such places are called the forbidden positions for the said objects.

The no of ways of carrying out this task is given by the following rule:

$$\bar{N} = S_0 - S_1 + S_2 - S_3 + \ldots + (-1)^n S_n.$$

where $S_0 = n!$, $S_K = (n-K)! \times r_K$ for $K = 1, 2, \ldots n$.

where r_K is the coefficient of x^K in the rook polynomial of the board of m rows & n columns whose squares represent the forbidden places.

Problems :-

1) Consider the board

1	2
3	4

find the rook polynomial

Soln :- Here $n = m = 4$.

The positions of 2 non-capturing rooks are : (1, 4) (2, 3)

$\therefore$ $r_2 = 2$.

The board has no positions for more than 2 non-capturing rooks. $\therefore$ $r_3 = r_4 = 0$.

$\therefore$ Rook polynomial is $r(C, x) = 1 + 4x + 2x^2$.

2)

1	2	3
4		5

Here $n = m = 5$

Positions of 2 non-capturing rooks are : (1, 5) (2, 4) (2, 5) (3, 4)

$\therefore$ $r_2 = 4$.

$r_3 = r_4 = r_5 = 0$.

Rook polynomial is $r(C, x) = 1 + 5x + 4x^2$.

3)

	1	2
	3	4
5	6	7

Here $n = m = 7$.

Positions of 2-non capturing rooks are:

(1, 4) (1, 5) (1, 7) (2, 3) (2, 5) (2, 6) (3, 5) (3, 7) (4, 5) (4, 6) $\therefore$ $r_2 = 10$.

~~$r_3 = r_4 = \ldots = r_7 = 0$.~~

Positions of 3 non-capturing rooks are :

(1, 4, 5) (2, 3, 5) $\therefore$ $r_3 = 2$.

$r_4 = \ldots = r_7 = 0$.

Rook polynomial is $r(C, x) = 1 + 7x + 10x^2 + 2x^3$.

4)

1	2	3
4		5
6	7	8

Here $n = m = 8$.

Positions of 2 non capturing rooks are :

(1, 5) (1, 7) (1, 8) (2, 4) (2, 5) (2, 6) (2, 8) (3, 4) (3, 6) (3, 7) (4, 7) (4, 8) (5, 6) (5, 7) $\Rightarrow$ $r_2 = 14$.

Positions of 3 non-capturing rooks are : (1, 5, 7) (2, 4, 8) (2, 5, 6) (3, 4, 7)

$\therefore$ $r_3 = 4$, $r_4 = \ldots r_8 = 0$. Rook poly is

$r(C, x) = 1 + 8x + 14x^2 + 4x^3$.

5) Find the rook polynomial for the 2x2 board by using expansion formula:

Soln :.

1 (*)	2
3	4

4

D

	2
3	4

E .

$r(D,x) = 1+x$.

for the board E, $n=3$, $r_2 = 1$ $[\because (2,3)]$

$r(E,x) = 1+3x+x^2$.

$\therefore$ Rook polynomial is $r(C,x) = x\,r(D,x) + r(E,x)$

$r(C,x) = x(1+x) + 1+3x+x^2$

$= x+x^2+1+3x+x^2$

$= \underline{1+4x+2x^2}$.

※.※

6) Find the rook polynomial for 3X3 board using the Expansion formula.

Soln :-

1	2	3
4	5 (*)	6
7	8	9

1	3
7	9

D

1	2	3
4		6
7	8	9

E .

for the board 'D',

$r_1 = 4$, $r_2 = 2$ $[\because (1,9)\,(3,7)]$, $r_3 = r_4 = 0$.

$\therefore r(D,x) = 1+4x+2x^2$.

for the board 'E',

$r_1 = 8$, $r_2 = 14$, $r_3 = 4$, $r_4 = \ldots r_8 = 0$. (Problem ④)

$\therefore r(E,x) = 1+8x+14x^2+4x^3$

Thus Rook polynomial for 3x3 board is

$r(C,x) = x\,r(D,x) + r(E,x) = x(1+4x+2x^2) + 1+8x+14x^2+4x^3$

$= x+4x^2+2x^3+1+8x+14x^2+4x^3$

$= \underline{1+9x+18x^2+6x^3}$

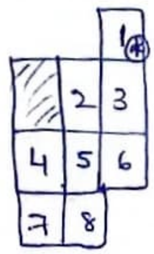
Verification :- we have for $n \times n$ board, Expansion formula is

$$r(C_{n\times n}, x) = 1 + \binom{n}{1}^2 x + 2! \times \binom{n}{2}^2 x^2 + 3! \times \binom{n}{3}^3 x^3$$

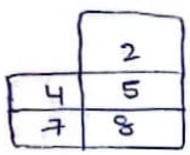
$$\therefore r(C_{3\times 3}, x) = 1 + \binom{3}{1}^2 x + 2! \times \binom{3}{2}^2 x^2 + 3! \times \binom{3}{3}^3 x^3$$

$$= 1 + 9x + 18x^2 + 6x^3$$

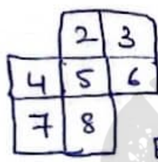
7) Using Expansion formula, obtain the rook polynomial for the board 'C' shown below :



Soln :-



D



E

for the board 'D',

$r_1 = n = 5$. Positions for 2 non-capturing rooks are :

(2,4) (2,7) (4,8) (5,7) $\therefore r_2 = 4$. $r_3 = 0 = r_4 = r_5$.

$\therefore$ $r(D, x) = 1 + 5x + 4x^2$

for the board 'E',

$r_1 = n = 7$, Positions for 2 non-capturing rooks are :

(2,4) (2,6) (2,7) (3,4) (3,5) (3,7) (3,8) (4,8) (5,7) (6,7) (6,8)

$\therefore$ $r_2 = 11$. Positions for 3 non-capturing rooks are :

(3,5,7) (2,6,7) (3,4,8) $\therefore r_3 = 3$.

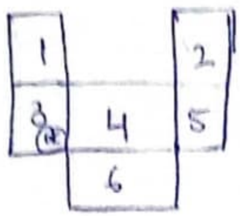
$r_4 = \dots = r_7 = 0$.

$\therefore$ $r(E, x) = 1 + 7x + 11x^2 + 3x^3$.

$$\therefore r(C, x) = x\, r(D, x) + r(E, x) = x\{1 + 5x + 4x^2\} + 1 + 7x + 11x^2 + 3x^3$$

$$= 1 + 8x + 16x^2 + 7x^3$$

8) Using Expansion formula, find the rook polynomial for the board C as shown below:

C : 

Soln :-

D :  E : 

In D, $r_1 = n = 2$, $r_2 = 1$ [$\because$ (2, 6)]

$\therefore \quad r(D, x) = 1 + 2x + x^2.$

In E, $r_1 = n = 5$. The positions for 2 non-capturing rooks are : (1,4) (1,5) (1,6) (2,4) (2,6) (5,6)

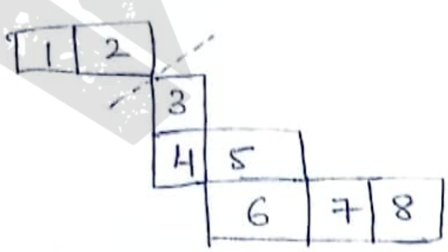
$\therefore \quad r_2 = 6.$

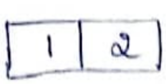
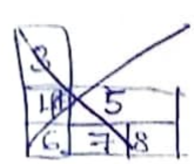
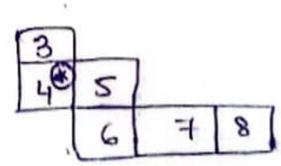
Also $r_3 = 1$ [$\because$ (1,5,6)]

$\therefore \quad r(E, x) = 1 + 5x + 6x^2 + x^3.$

Thus
$$\begin{aligned} r(C, x) &= x\, r(D, x) + r(E, x) \\ &= x\left[1 + 2x + x^2\right] + \left[1 + 5x + 6x^2 + x^3\right] \\ &= x + 2x^2 + x^3 + 1 + 5x + 6x^2 + x^3 \\ &= 1 + 6x + 8x^2 + 2x^3. \end{aligned}$$

9) obtain the rook polynomial for the foll chess board

C : 

C_1 :  C_2 :  

In C_1, $r_1 = 2$, $r_2 = 0$.

$\therefore \quad r(C_1, x) = 1 + 2x.$

In C_2, let us mark square 4 as (*) then the boards D & E are as below:

In D, $r_1 = 3$. $r_2 = r_3 = 0$.

$\therefore\ r(D,x) = 1+3x$.

In E, $r_1 = 5$. Positions for 2 non-capturing rooks are:

(3,5) (3,6) (3,7) (3,8) (5,7) (5,8). $\therefore\ r_2 = 6$

Positions for 3 non-capturing rooks are:

(3,5,7) (3,5,8) $\therefore\ r_3 = 2$.

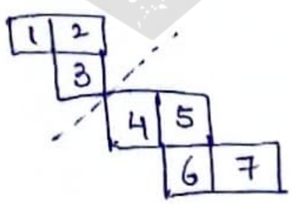
Thus $r(E,x) = 1+5x+6x^2+2x^3$.

$$\therefore\ r(C_2,x) = x\, r(D,x) + r(E,x)$$
$$= x(1+3x) + 1+5x+6x^2+2x^3$$
$$= 1+6x+9x^2+2x^3.$$

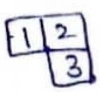
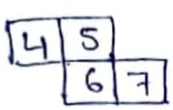
Thus
$$r(C,x) = r(C_1,x) \times r(C_2,x)$$
$$= (1+2x)(1+6x+9x^2+2x^3)$$
$$= 1+6x+9x^2+2x^3+2x+12x^2+18x^3+4x^4$$
$$= 1+8x+21x^2+20x^3+4x^4.$$

10) Find the rook polynomial for the board C:

C :



Soln :-

C_1 :  C_2 : 

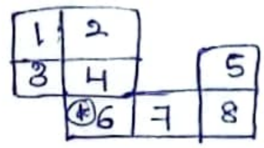
In C_1, $n = r_1 = 3$. $r_2 = 1$. $\Rightarrow r(C_1,x) = 1+3x+x^2$.

In C_2, $n = r_1 = 4$, $r_2 = 3$ $\Rightarrow r(C_2,x) = 1+4x+3x^2$.

$\therefore \quad r(C,x) = r(C_1,x) \times r(C_2,x)$

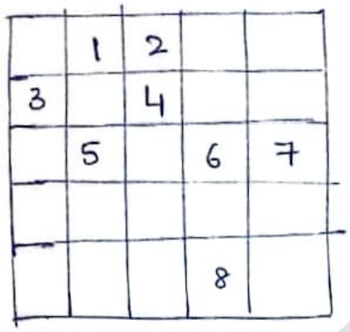
$$\Rightarrow r(C,x) = (1+3x+x^2)(1+4x+3x^2)$$
$$= 1+4x+3x^2+3x+12x^2+9x^3+x^2+4x^3+3x^4$$
$$= 1+7x+16x^2+13x^3+3x^4$$

11) Find the rook polynomial of: the

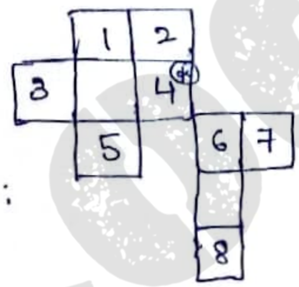


$\rightarrow r(C,x) = 1+8x+16x^2+5x^3$.

12). Find the rook poly for the board made up of the shaded squares in the figure.



C :

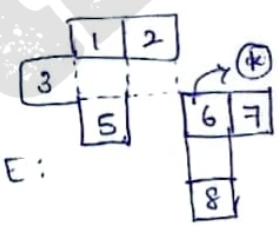


Mark the square 6 as ⊛ & proceed (see prob 15)

Soln :- Let us mark the ~~board~~ square numbered 4 as ⊛, then the boards D & E are as below:

D :

E :



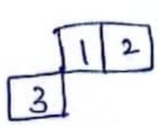
In D, $r_1 = 5$, $r_2 = 5$ [(1,6) (1,7) (1,8) (5,8) (7,8)]

$r_3 = 1$ [(1,7,8)]

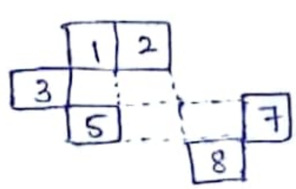
$\therefore \quad r(D,x) = 1+5x+5x^2+x^3$.

In E, let us mark the ~~board~~ square 6 as ⊛, then the boards E_1 & E_2 are as below:

E_1 :



E_2 :



In E_1, $r_1 = 3$, $r_2 = 2$, $r_3 = 0$.

$\therefore \quad r(E_1, x) = 1 + 3x + 2x^2$.

The board E_2 can be split into 2 disjoint boards as follows:

E_3  E_4 :

In E_3, $r_1 = 5$, $r_2 = 7$ $\left[(1,3)(1,7)(2,3)(2,5)(2,7)(3,5)(3,7)\right]$

$r_3 = 3$ $\left[(1,3,7)(2,3,5)(2,3,7)\right]$

$\therefore \quad r(E_3, x) = 1 + 5x + 7x^2 + 3x^3$.

Also $r(E_4, x) = 1 + x$.

$$\therefore \quad r(E_2, x) = (1+5x+7x^2+3x^3)(1+x)$$
$$= 1 + x + 5x + 5x^2 + 7x^2 + 7x^3 + 3x^3 + 3x^4$$
$$= 1 + 6x + 12x^2 + 10x^3 + 3x^4.$$

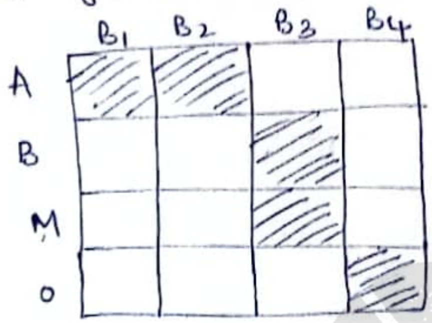
Thus $r(E, x) = x\, r(E_1, x) + r(E_2, x)$
$$= x\left[1 + 3x + 2x^2\right] + 1 + 6x + 12x^2 + 10x^3 + 3x^4$$
$$r(E, x) = 1 + 7x + 15x^2 + 12x^3 + 3x^4.$$

Hence $r(C, x) = x\, r(D, x) + r(E, x)$

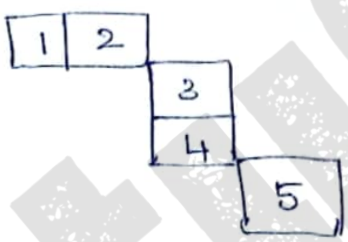
$$\Rightarrow r(C, x) = x\left[1 + 5x + 5x^2 + x^3\right] + 1 + 7x + 15x^2 + 12x^3 + 3x^4$$
$$= 1 + 8x + 20x^2 + 17x^3 + 4x^4$$

13) An apple, a banana, a mango and an orange are to be distributed to 4 boys B_1, B_2, B_3, B_4. The Boys B_1 & B_2 do not want to have an apple, the boy B_3 does not want banana (or) mango, and B_4 refuses orange. In how many ways can the distribution be made so that no boy is displeased.

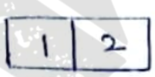
Soln :- Let us construct a board (table) in which rows represent apple, banana, mango & orange respectively, and columns represents Boys B_1, B_2, B_3, B_4 respectively. Shaded square represents forbidden positions in distribution.

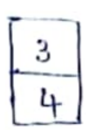


Consider the board 'C' consisting of shaded square:



'C' can be divided into 3 disjoint subparts C_1, C_2, C_3 as shown below:

C_1 : 

C_2 : 

C_3 : 

$\therefore\ r(C, x) = r(C_1, x) \times r(C_2, x) \times r(C_3, x) \longrightarrow (1)$

In C_1, $r_1 = n = 2$, $r_2 = 0 \Rightarrow r(C_1, x) = 1 + 2x$.

$|||^{ly}$ $r(C_2, x) = 1 + 2x$.

$r(C_3, x) = 1 + x$.

$(1) \Rightarrow\ r(C, x) = (1+2x)^2(1+x)$

$= (1 + 4x^2 + 4x)(1+x)$

$$r(c,x) = 1+x+4x^2+4x^3+4x+4x^2$$

$$r(C,x) = 1+5x+8x^2+4x^3 .$$

Thus for the board C,

$r_1 = 5, \quad r_2 = 8, \quad r_3 = 4 .$

we have $S_0 = 4! = 24$, $S_1 = (4-1)! \times r_1 = 6\times 5 = 30$.

$S_2 = (4-2)! \times r_2 = 2\times 8 = 16$, $S_3 = (4-3)! \times r_3 = 1\times 4 = 4$

$$\therefore \quad \bar{N} = S_0 - S_1 + S_2 - S_3$$

$$= 24 - 30 + 16 - 4$$

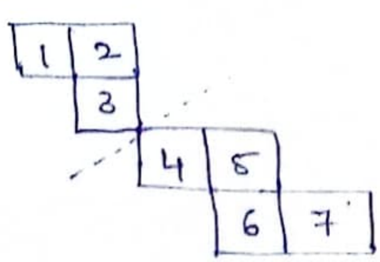
$$\bar{N} = 6 .$$

$\therefore$ There are 6 ways of distributing fruits so that no one is displaced.

14) Four persons P_1, P_2, P_3, P_4 who arrive late for a dinner party find that only one chair at each of 5 tables T_1, T_2, T_3, T_4, T_5 is vacant. P_1 will not sit at T_1 or T_2, P_2 will not sit at T_2, P_3 will not sit at T_3 or T_4 and P_4 will not sit at T_4 or T_5. Find the no of ways they can occupy the vacant chairs.

Soln :- Consider the board which represents the given situation. Let the rows represents 4 persons P_1, P_2, P_3, P_4, and columns represents the tables $T_1, \dots T_5$. The shaded squares represent the forbidden positions.

consider the board 'C' consisting of shaded squares :



Refer Problem (10).

$$r(C,x) = 1 + 7x + 16x^2 + 13x^3 + 3x^4$$

Thus for the board 'C',

$$r_1 = 7, \quad r_2 = 16, \quad r_3 = 13, \quad r_4 = 3.$$

$\therefore$ Required no of ways $= S_0 - S_1 + S_2 - S_3 + S_4$

$$= 5! - (5-1)! \times r_1 + (5-2)! \times r_2 - (5-3)! \times r_3 + (5-4)! \times r_4$$

$$= 5! - 24 \times 7 + 6 \times 16 - 2 \times 13 + 1 \times 3$$

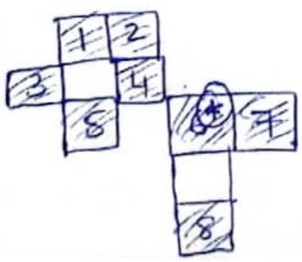
$$= 120 - 168 + 96 - 26 + 3$$

$$= 25.$$

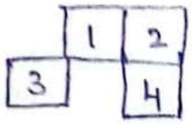
15) A girl student has sarees of 5 different colors blue, green, red, white & Yellow. on Monday, she does not wear green, on Tuesday blue or red, on wednesday blue or green, on Thursday red or yellow, on friday red. In how many ways can she dress without repeating a color from Monday to friday.

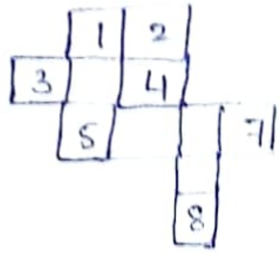
Soln:- The situation can be represented by the foll board where shaded squares represents the forbidden positions.

Consider the board C containing the shaded squares :



Mark the square 6 as (*), so that the boards D, E are as shown below :

D: 

E: 

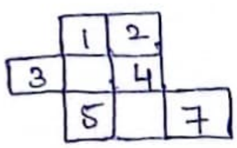
$\therefore \quad r(C,x) = x\, r(D,x) + r(E,x) \rightarrow (1)$

for the board D,

$r_1 = n = 4. \qquad r_2 = 3.$

$\therefore \quad r(D,x) = 1 + 4x + 3x^2.$

The board E can be divided into 2 disjoint subparts E_1 & E_2 as below :

E_1 : 

E_2 : $\boxed{8}$

for E_1, $r_1 = n = 6$. $r_2 = 10$ [$\because$ positions of non-capturing rooks are (1,3) (1,4) (1,7) (2,3) (2,5) (2,7) (3,5) (3,7) (4,5) (4,7)]

Positions of 3 non-capturing rooks are (1,3,7) (1,4,7) (2,3,5) (2,3,7) $\Rightarrow$ $r_3 = 4$

$\therefore \quad r(E_1,x) = 1 + 6x + 10x^2 + 4x^3.$

& $r(E_2,x) = 1 + x.$

$$\begin{aligned}\therefore \quad r(E,x) &= r(E_1,x) \times r(E_2,x) \\ &= (1+6x+10x^2+4x^3)(1+x) \\ &= 1+6x+10x^2+4x^3+x+6x^2+10x^3+4x^4 \\ &= 1+7x+16x^2+14x^3+4x^4.\end{aligned}$$

Thus (1) $\Rightarrow$ $r(C,x) = x(1+4x+3x^2) + 1+7x+16x^2+14x^3+4x^4$

$= 1+8x+20x^2+17x^3+4x^4.$

for the board C,

$r_1 = 8$, $r_2 = 20$, $r_3 = 17$, $r_4 = 4$.

$\therefore$ Required no., $\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4$

$$= 5! - (5-1)! \times 8 + (5-2)! \times 20 - (5-3)! \times 17 + (5-4)! \times 4$$

$$= 18.$$

16) Five Teachers $T_1, \ldots T_5$ are to be made class teachers for 5 classes $C_1, C_2 \ldots C_5$ - one teacher for each class. T_1 & T_2 do not wish to become class teachers for C_1 or C_2, T_3 & T_4 for C_4 or C_5 and T_5 for C_3 or C_4 or C_5. In how many ways can the teachers be assigned the work without displeasing any teacher.

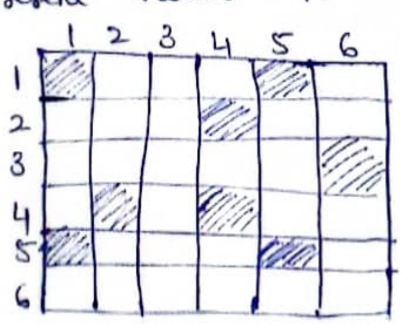
HW

$\rightarrow$ $r(c, x) = 1 + 11x + 40x^2 + 56x^3 + 28x^4 + 4x^5$.

$\therefore$ $\bar{N} = 8$.

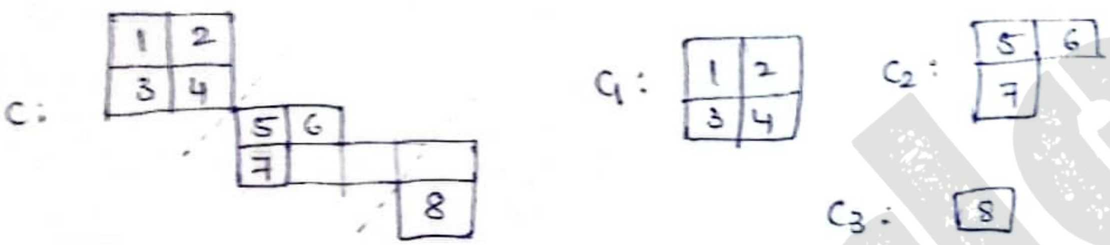
17) A pair of dice one red & other green is rolled six times. Find the probability that we obtain 6 values on both the red die & green die under the restriction that the ordered pairs (1,1) (1,5) (2,4) (3,6) (4,2) (4,4) (5,1) & (5,5) do not occur. [Here an ordered pair (a,b) indicates 'a' on the red die & 'b' on the green]

Soln:- Consider the foll table representing the situation in which rows represent the value appearing on red die & columns represent values appearing on green die.



Let us redraw the table as follows:-
(Since all shaded region are scattered, we put them together as below)

The board 'C' is as shown below:



$$r(C,x) = r(C_1,x) \times r(C_2,x) \times r(C_3,x) \rightarrow (1)$$

$$r(C_1,x) = 1+4x+2x^2$$
$$r(C_2,x) = 1+3x+x^2$$
$$r(C_3,x) = 1+x$$

$$(1) \Rightarrow \quad r(C,x) = (1+4x+2x^2)(1+3x+x^2)(1+x)$$
$$= (1+4x+2x^2)(1+3x+x^2+x+3x^2+x^3)$$
$$= (1+4x+2x^2)(1+4x+4x^2+x^3)$$
$$= 1+4x+4x^2+x^3+4x+16x^2+16x^3+4x^4 + 2x^2+8x^3+8x^4+2x^5$$
$$= 1+8x+22x^2+25x^3+12x^4+2x^5$$

$\therefore$ $r_1 = 8$, $r_2 = 22$, $r_3 = 25$, $r_4 = 12$, $r_5 = 2$.

$\therefore$ $\bar{N} = S_0 - S_1 + S_2 - S_3 + S_4 - S_5 = 160$.

$\therefore$ No of favourable outcomes $= 6! \times 160$.

The sample space 'S' contains the set of all outcomes such that the 8 pairs do not occur.

$\therefore$ $|S| = (36-8)^6 = 28^6$.

$$\therefore \text{ Required probability} = \frac{6! \times 160}{(28)^6}.$$

$$= 2.391 \times 10^{-4}$$

Recurrence Relations

* First order Recurrence relations :

A I order recurrence relation with constant coefficients is of the form

$$a_n = c\, a_{n-1} + f(n) \longrightarrow (1) \quad \text{for } n \geq 1$$

where $c \rightarrow$ a known constant.

$f(n) \rightarrow$ a known function of f.

If $f(n) = 0$, then (1) is called a homogenous recurrence relation ; otherwise it is called a non-homogenous recurrence relation.

Note :- 1) The General sol^n of a homogenous R R is $a_n = c^n a_0$ for $n \geq 1$

2) The General sol^n of a non-homogenous R.R of order 1 is given by $a_n = c^n a_0 + \sum_{k=1}^{n} c^{n-k} f(k)$, for $n \geq 1$.

Problems :-

1) Solve the recurrence relation $a_n = 7a_{n-1}$, where $n \geq 1$ given that $a_2 = 98$.

Sol^n :- $a_n = 7\, a_{n-1} \longrightarrow (1)$ is a homogenous 1st order recurrence relation.

General sol^n is given by

$$a_n = c^n a_0 .$$

$$a_n = 7^n a_0 \longrightarrow (2)$$

for $n = 2$; $a_2 = 7^2 a_0 \Rightarrow a_2 = 49 a_0$

$\Rightarrow 98 = 49 a_0 \Rightarrow \boxed{a_0 = 2}$

Sub in (2), $\boxed{a_n = 2 \cdot 7^n}$ is the General sol^n.

2) Solve the recurrence relation $a_n = n\, a_{n-1}$ for $n \geq 1$ given that $a_0 = 1$.

Sol^n:.
$$a_n = n\, a_{n-1}$$

$n=1$; $a_1 = 1 \times a_0$

$n=2$; $a_2 = 2 \times a_1 = (2 \times 1) \times a_0$

$n=3$; $a_3 = 3 \times a_2 = (3 \times 2 \times 1) \times a_0$.

$n=4$; $a_4 = 4 \times a_3 = (4 \times 3 \times 2 \times 1) \times a_0$. and so on.

$\therefore$ General sol^n is

$$a_n = n!\, a_0 \quad \text{for } n \geq 1.$$

using $a_0 = 1 \Rightarrow \boxed{a_n = n!}$ is the required sol^n.

3) Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 3^n$ for $n \geq 1$, given that $a_0 = 2$.

Sol^n:- Given $a_n = 3\, a_{n-1} + (5 \times 3^n) \longrightarrow (1)$ is a non-homo. relation with $c = 3$, $f(n) = 5 \times 3^n$.

General sol^n is given by

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k)$$

$$a_n = 3^n a_0 + 3^{n-1} f(1) + 3^{n-2} f(2) + \dots + 3^0 f(n).$$

$$\Rightarrow a_n = 3^n \times 2 + 3^{n-1}(5 \times 3^1) + 3^{n-2}(5 \times 3^2) + \dots + 3^0 \times (5 \times 3^n)$$

$$= 2 \times 3^n + 5 \times [3^n + 3^n + \dots + 3^n] \text{ (n times)}$$

$$= 2 \times 3^n + 5 \times n \times 3^n$$

$\boxed{a_n = 3^n(2 + 5n)}$ is the required sol^n.

4) Solve the recurrence relation $a_n - 3a_{n-1} = 5 \times 7^n$, for $n \geq 1$, given that $a_0 = 2$.

P.T.O.

Sol^n :. Given : $a_n = 3a_{n-1} + (5 \times 7^n) \rightarrow (1)$ is a non-homo. recurrence relation with $c = 3$, $f(n) = 5 \times 7^n$.

The general sol^n is given by

$$a_n = 3^n a_0 + \sum_{k=1}^{n} 3^{n-k} f(k).$$

$$= 3^n \times 2 + \sum_{k=1}^{n} 3^{n-k} \times (5 \times 7^k)$$

$$= 2 \times 3^n + (5 \times 3^n) \sum_{k=1}^{n} 3^{-k} \cdot 7^k.$$

$$= 2 \times 3^n + (5 \times 3^n) \sum_{k=1}^{n} \left(\frac{7}{3}\right)^k.$$

$$= 2 \times 3^n + (5 \times 3^n)\left[\frac{7}{3} + \left(\frac{7}{3}\right)^2 + \dots + \left(\frac{7}{3}\right)^n\right]$$

$$= 2 \times 3^n + (5 \times 3^n) \times \frac{7}{3}\left[1 + \left(\frac{7}{3}\right) + \left(\frac{7}{3}\right)^2 + \dots + \left(\frac{7}{3}\right)^{n-1}\right]$$

$$a + ar + ar^2 + \dots = \frac{a(r^n - 1)}{r - 1} ; r > 1$$

here $a = 1$, $r = 7/3$

$$\therefore a_n = 2 \times 3^n + (35 \times 3^{n-1}) \left[\frac{\left(\frac{7}{3}\right)^n - 1}{\left(\frac{7}{3} - 1\right)}\right]$$

$$= (2 \times 3^n) + (35 \times \cancel{3^{n-1}}) \times \frac{\cancel{3}}{4}\left[\frac{7^n - 3^n}{\cancel{3^n}}\right]$$

$$= (2 \times 3^n) + \left(\frac{35}{4}\right) (7^n - 3^n)$$

$$= 3^n\left[2 - \frac{35}{4}\right] + \frac{35}{4} \cdot 7^n$$

$$= -3^n \times \frac{27}{4} + \frac{5 \times 7 \times 7^n}{4}$$

$$= -\frac{1}{4} \cdot 3^{n+3} + \frac{5}{4} \cdot 7^{n+1}$$

$$\boxed{a_n = \frac{1}{4}\left[5 \times 7^{n+1} - 3^{n+3}\right]}$$ is the required sol^n.

5) Solve the recurrence relation

$a_n = 2a_{n/2} + (n-1)$ for $n = 2^k$, $k \geq 1$, given $a_1 = 0$.

Soln:. $a_n = 2a_{n/2} + (n-1)$

$\Rightarrow a_n - 2a_{n/2} = (n-1)$.

we obtain the following successive eqns

$a_{n/2} - 2a_{n/4} = \left(\frac{n}{2} - 1\right)$

$a_{n/4} - 2a_{n/8} = \left(\frac{n}{4} - 1\right)$

$\vdots$

$a_{n/2^{k-1}} - 2a_{n/2^k} = \left(\frac{n}{2^{k-1}} - 1\right)$

These can be written as

$a_n - 2a_{n/2} = (n-1)$.

$2a_{n/2} - 2^2 a_{n/4} = (n-2)$

$2^2 a_{n/4} - 2^3 a_{n/8} = (n-2^2)$

$\vdots$

$2^{k-1} a_{n/2^{k-1}} - 2^k \cdot a_{n/2^k} = (n - 2^{k-1})$.

adding these, we get

$a_n - 2^k \cdot a_{n/2^k} = (n-1) + (n-2) + (n-2^2) + \dots + (n - 2^{k-1})$

since $n = 2^k$, $a_{n/2^k} = a_1 = 0$ (given)

$\therefore a_n = (n + n + \dots + n)_{k \text{ times}} - (1 + 2 + 2^2 + \dots + 2^{k-1})$

$a + ar + ar^2 + \dots = \frac{a(r^k - 1)}{r-1}$, $r > 1$

$a = 1$, $r = 2$

$= kn - \frac{(1)(2^k - 1)}{2-1}$

$= kn - (2^k - 1) = kn - (n-1)$ $(\because n = 2^k)$

$= 1 + (k-1)n$

$\boxed{a_n = 1 + [\log_2 n - 1]n}$

$\Big(\because$ $2^k = n$.

$\log 2^k = \log n$

$k \cdot \log 2 = \log n$.

$k = \frac{\log_e n}{\log_e 2} = \log_2 n \Big)$

6) Find the recurrence relation and the initial condⁿ for
QP the sequence 0, 2, 6, 12, 20, 30, 42 .... Hence find the general term of the sequence.

Solⁿ:-

Given $a_0 = 0$, $a_1 = 2$, $a_2 = 6$, $a_3 = 12$, $a_4 = 20$ ..

Consider $a_1 - a_0 = 2 - 0 = 2 = 2\times 1$

$a_2 - a_1 = 6 - 2 = 4 = 2\times 2$

$a_3 - a_2 = 12 - 6 = 6 = 2\times 3$

$a_4 - a_3 = 20 - 12 = 8 = 2\times 4$

$\vdots$

$a_n - a_{n-1} = 2\times n$ is the R.R with the initial condⁿ $a_0 = 0$.

adding all these,

$$a_n - a_0 = (2\times 1) + (2\times 2) + (2\times 3) + (2\times 4) + \dots + (2\times (n-1)) + (2\times n)$$

$$a_n - 0 = 2[1 + 2 + 3 + \dots + n]$$

$$a_n = 2\,\frac{n(n+1)}{2}$$

$$\boxed{a_n = n(n+1)}$$

7) The number of virus affected files in a system is 1000 (to
QP start with) and this increases 250% every 2 hours. Use a recurrence relation to determine the no. of virus affected files in the system after one day.

Solⁿ:- Let $a_0 = 1000$

Let a_n denote the no. of virus affected files after $2n$ hours.

It is given that the no. increases by 250% every 2 hours.

$\therefore$ $a_1 = a_0 + 250\%\, a_0$

$a_2 = a_1 + 250\%\, a_1$

$\vdots$

$a_n = a_{n-1} + 250\%\, a_{n-1}$

$\therefore \quad a_n = a_{n-1}\left[1+250\%\right]$

$= a_{n-1}\left[1+\frac{250}{100}\right]$

$= a_{n-1}(1+2.5)$

$\boxed{a_n = 3.5\,a_{n-1}} \quad \forall\ n\geq 1.$

This is the recurrence relation for the no. of virus affected files.

$\therefore$ General soln of the recurrence relation is given by

$a_n = c^n a_0.$

$a_n = (3.5)^n \times 1000.$

This gives the no. of virus affected files after $2n$ hours.

$\therefore$ No. of virus affected files after 24 hrs (1 day) (when $n=12$) is

$a_{12} = (3.5)^{12} \times 1000. = 3379220508.$

8) A person invests Rs. 10,000 at 10.5% interest (per year) compounded monthly. find and solve the recurrence relation for the value of the investment at the end of n months. what is the value of the investment at the end of the I year? How long will it take to double the investment?

Soln :- Let a_0 denote the initial investment.

Let $a_1, a_2, \ldots a_n$ denote the investments after 1, 2, 3 .... n months respectively.

Given : annual rate of interest = 10.5%.

$\therefore$ Monthly rate of interest $= \frac{10.5\%}{12} = 0.875\%.$

Thus $a_0 = 10000$

$a_1 = a_0 + (0.875\%)\,a_0.$

$a_2 = a_1 + (0.875\%)\,a_1$

$\vdots$

$a_n = a_{n-1} + (0.875\%)\,a_{n-1}$

$a_n = a_{n-1}\left[1+0.875\%\right]$

$$\Rightarrow a_n = a_{n-1}\left[1 + \frac{0.875}{100}\right]$$

$$\boxed{a_n = 1.00875\, a_{n-1}}, \quad \forall\, n \geq 1$$

This is the Recurrence relation at the end of 'n' months.

$\therefore$ General Soln of the recurrence relation is given by

$$a_n = c^n a_0$$

$$a_n = (1.00875)^n \times 10000$$

$\therefore$ Investment at the end of first year is $(n = 12)$

$$a_{12} = (1.00875)^{12} \times 10000$$

$$= 11102.03$$

$$a_{12} \cong 11,102$$

Next, to find n given that

$$a_n = 2a_0.$$

$$\Rightarrow (1.00875)^n \times \cancel{10000} = 2 \times \cancel{10000}$$

$$\Rightarrow (1.00875)^n = 2$$

$$\Rightarrow n \log_e(1.00875) = \log_e 2$$

$$\Rightarrow n = \frac{\log_e 2}{\log_e(1.00875)} = 79.56.$$

$$\boxed{n \cong 80}.$$

Thus the investment will be doubled in about 80 months time ie 6 years and 8 months.

9) A bank pays a certain % of annual interest on deposits, compounding the interest once in 3 months. If a deposit doubles in 6 years and 6 months, what is the annual % of interest paid by the bank?.

Soln:. Let the annual rate of interest be x%.

$\therefore$ Quarterly rate of interest is $\left(\frac{x}{4}\right)$%.

Let a_0 be the initial deposit and a_n be the deposit ~~after~~ at the end of n^{th} quarter.

$a_1 = a_0 + \left(\frac{x}{4}\%\right)a_0$

$a_2 = a_1 + \left(\frac{x}{4}\%\right)a_1$

$a_n = a_{n-1} + \left(\frac{x}{4}\%\right)a_{n-1}$

$= a_{n-1}\left[1 + \frac{x}{4}\%\right]$

$\boxed{a_n = a_{n-1}\left(1+\frac{x}{400}\right)} \rightarrow (1)$ is the recurrence relation

General solⁿ of (1) is

$a_n = c^n a_0$

$\Rightarrow a_n = \left(1+\frac{x}{400}\right)^n a_0$

Given that the deposit doubles in 6 yrs, 6 months (ie 78 months) ie deposit doubles in 26 quarters $\left(\because \frac{78}{3} = 26\right)$

ie $n = 26$.

we have $a_n = 2a_0$.

ie $a_{26} = 2a_0$.

$\Rightarrow \left(1+\frac{x}{400}\right)^{26} \cancel{a_0} = 2\cancel{a_0}$.

$\Rightarrow 26 \log_e\left(1+\frac{x}{400}\right) = \log_e 2$

$\Rightarrow \log_e\left(1+\frac{x}{400}\right) = 0.0266595$

$\Rightarrow 1+\frac{x}{400} = e^{0.0266595} = 1.027$

$\Rightarrow \frac{x}{400} = 0.027 \Rightarrow \boxed{x = 10.8}$

Thus the annual rate of interest paid by the bank is 10.8% (compounding the interest once in 3 months).

10) A bank pays 6% interest compound quarterly. If Laura invests Rs. 100 then how many months must she wait for her money to double?

HINT :- 3 months – 6% interest
1 months – ?
$\frac{1 \times 6\%}{3} = 2\%$.

$a_0 = 100$
$a_n = a_{n-1}(1+2\%)$
$a_n = 1.02\, a_{n-1}$

$a_n = 2a_0$.
$\Rightarrow n = 35$ months.

Second Order homogenous Recurrence Relation

A second order homogenous recurrence relation is of the form $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0 \quad \forall n \geq 2 \rightarrow (1)$

where C_n, C_{n-1}, C_{n-2} are real constants.

The auxilary equation of eq^n (1) is given by

$$C_n K^2 + C_{n-1} K + C_{n-2} = 0$$

Suppose K_1 and K_2 are the roots of A.E

Case (i). If K_1 and K_2 are real and distinct, then general soln of eq^n (1) is given by

$$a_n = A K_1^n + B K_2^n; \text{ where } A \;\&\; B \text{ are arbitrary constants}$$

Case (ii): If $K_1 = K_2 = K$, then general soln of eq^n (1) is given by

$$a_n = (A + Bn) K^n.$$

Case (iii): If K_1 and K_2 are imaginary i.e if $K_1 = p + iq$, and $K_2 = p - iq$, then the general soln of eq^n (1) is given by

$$a_n = r^n [A \cos n\theta + B \sin n\theta]. \quad \text{where } r = \sqrt{p^2+q^2}$$
$$\theta = \tan^{-1}\left(\frac{q}{p}\right).$$

Solve the following Recurrence relations:

1) $a_n + a_{n-1} - 6 a_{n-2} = 0 \quad \forall n \geq 2$ given $a_0 = -1, a_1 = 8$.
$\rightarrow (1)$

soln:- comparing with $C_n a_n + C_{n-1} a_{n-1} + C_{n-2} a_{n-2} = 0$, we have $C_n = 1, C_{n-1} = 1, C_{n-2} = -6$.

A.E is $C_n K^2 + C_{n-1} K + C_{n-2} = 0$.

$\Rightarrow K^2 + K - 6 = 0$

$\Rightarrow (K+3)(K-2) = 0$.

$\Rightarrow$ Roots are $\boxed{K_1 = -3, K_2 = 2}$ real & distinct roots.

General soln of (1) is given by

$$a_n = A.(-3)^n + B.2^n \rightarrow (2).$$

Given $a_0 = -1$, $a_1 = 8$.

sub $n = 0$ in (2),

$a_0 = A(-3)^0 + B(2)^0$

$-1 = A + B \rightarrow$ (3)

sub $n=1$ in (2)

$a_1 = A(-3)^1 + B(2)^1$

$8 = -3A + 2B \rightarrow$ (4)

Solving (3) & (4), $\boxed{A = -2, B = 1}$

Sub in (2), $a_n = -2(-3)^n + 1\cdot(+2)^n$

ie $a_n = 2^n - 2(-3)^n$

2) $2a_n = 7a_{n-1} - 3a_{n-2}$, $n \geq 2$, $a_0 = 2$, $a_1 = 5$.

Soln:- $2a_n - 7a_{n-1} + 3a_{n-2} = 0. \rightarrow$ (1).

AE: $2k^2 - 7k + 3 = 0$.

Roots are $k_1 = 3$, $k_2 = 1/2$. (real & distinct)

General soln is given by

$a_n = A\cdot 3^n + B\cdot\left(\frac{1}{2}\right)^n \rightarrow$ (2)

given $a_0 = 2$, $a_1 = 5$

sub $n=0$ in (2), $a_0 = A + B$.

$\Rightarrow 2 = A + B \rightarrow$ (3)

sub $n=1$ in (2), $a_1 = 3A + \frac{1}{2}B$

$\Rightarrow 5 = \frac{6A + B}{2}$ (or) $6A + B = 10 \rightarrow$ (4)

solving (3) & (4), $\boxed{A = 8/5}$, $\boxed{B = 2/5}$.

sub in (2), $a_n = \left(\frac{8}{5}\right)3^n + \left(\frac{2}{5}\right)\left(\frac{1}{2}\right)^n$

3) $a_n - 6a_{n-1} + 9a_{n-2} = 0$, $n \geq 2$, $a_0 = 5$, $a_1 = 12$.
$\hookrightarrow$ (1)

soln:- AE: $k^2 - 6k + 9 = 0$.

$\Rightarrow \boxed{K = 3, 3}$

Roots are real & repeated.

$\therefore$ General soln of ① is

$$a_n = (A + Bn)3^n \longrightarrow ②$$

Given $a_0 = 5$, $a_1 = 12$

sub $n = 0$ in ② $\Rightarrow a_0 = A \cdot 3^0$

$$\boxed{5 = A}$$

sub $n = 1$ in ② $\Rightarrow a_1 = (A + B)3^1$

$$12 = (5 + B)3$$
$$12 = 15 + 3B$$
$$3B = -3 \Rightarrow \boxed{B = -1}$$

sub in ①, $\boxed{a_n = (5 - n)3^n}$

4) $4a_n + 2a_{n-1} + a_{n-2} = 0$.

Soln: AE: $4K^2 + 2K + 1 = 0$.

$$K = \frac{-2 \pm \sqrt{4 - 16}}{8} = \frac{-2 \pm \sqrt{-12}}{8} = \frac{-2 \pm 2i\sqrt{3}}{8}$$

$$\boxed{K = \frac{-1 \pm \sqrt{3}i}{4}}$$

Roots are $K_1 = -\frac{1 + \sqrt{3}i}{4}$, $K_2 = -\frac{1 - \sqrt{3}i}{4}$ (imaginary roots)

comparing with $p \pm iq$, $p = -\frac{1}{4}$, $q = \frac{\sqrt{3}}{4}$.

$$\therefore r = \sqrt{p^2 + q^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}\left(\frac{\sqrt{3}/4}{-1/4}\right) = \tan^{-1}(-\sqrt{3}) = -60^\circ = -\pi/3$$

$\therefore$ General soln of ① is

$$a_n = r^n\left[A \cos n\theta + B \sin n\theta\right]$$

$$a_n = \left(\frac{1}{2}\right)^n \left[A \cos\left(-\frac{n\pi}{3}\right) + B \sin\left(-\frac{n\pi}{3}\right)\right].$$

5) $a_n = 2(a_{n-1} - a_{n-2})$, for $n \geq 2$ given that $a_0 = 1$ & $a_1 = 2$.

HW

Ans: $a_n = (\sqrt{2})^n\left[\cos\frac{n\pi}{4} + \sin\frac{n\pi}{4}\right]$.

6) $D_n = bD_{n-1} - b^2 D_{n-2}$ for $n \geq 3$, given $D_1 = b > 0$, $D_2 = 0$

Soln:- $D_n - bD_{n-1} + b^2 D_{n-2} = 0 \rightarrow (1)$

AE: $k^2 - bk + b^2 = 0$

$$k = \frac{b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{b \pm \sqrt{-3b^2}}{2} = \frac{b \pm i\sqrt{3}\,b}{2}$$

$\therefore k_1 = \frac{b}{2} + i\frac{\sqrt{3}b}{2}$ and $k_2 = \frac{b}{2} - \frac{i\sqrt{3}b}{2}$ (imaginary roots)

$\therefore$ General soln for D_n is

$$D_n = r^n [A \cos n\theta + B \sin n\theta] \rightarrow (2)$$

where A and B are arbitrary constants.

$p = \frac{b}{2}, q = \frac{\sqrt{3}b}{2}$

$$r = \sqrt{p^2 + q^2} = \sqrt{\frac{b^2}{4} + \frac{3b^2}{4}} = \sqrt{\frac{4b^2}{4}} = b.$$

$$\theta = \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}\left(\frac{\sqrt{3}b/2}{b/2}\right) = \tan^{-1}\sqrt{3} = \pi/3.$$

(2) $\Rightarrow D_n = b^n \left[A \cos \frac{n\pi}{3} + B \sin \frac{n\pi}{3}\right] \rightarrow (3)$ is the g. soln.

given $D_1 = b$, $D_2 = 0$.

Put $n = 1$ in (3), $D_1 = b\left[A \cos\frac{\pi}{3} + B \sin\frac{\pi}{3}\right]$

$$\cancel{b} = \cancel{b}\left[A \cdot \frac{1}{2} + B \cdot \frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow \boxed{1 = \frac{1}{2}A + \frac{\sqrt{3}}{2}B.} \rightarrow (4)$$

Put $n = 2$ in (3), $D_2 = b^2\left[A \cos\frac{2\pi}{3} + B \sin\frac{2\pi}{3}\right]$

$$0 = b^2\left[A \cos(180^\circ - 60^\circ) + B \sin(180^\circ - 60^\circ)\right]$$

$$\Rightarrow 0 = -A \cos 60^\circ + B \sin 60^\circ$$

$$\Rightarrow \boxed{0 = -\frac{1}{2} \cdot A + \frac{\sqrt{3}}{2} B} \rightarrow (5)$$

Solving (4) & (5), $\boxed{A = 1, \; B = 1/\sqrt{3}}$

Sub in (3),

$$D_n = b^n \left[\cos\frac{n\pi}{3} + \frac{1}{\sqrt{3}} \sin\frac{n\pi}{3}\right].$$

7) $F_{n+2} = f_{n+1} + f_n$ for $n \geq 0$, given $F_0 = 0$, $F_1 = 1$.

Soln:- Rewriting as $f_n = f_{n-1} + f_{n-2}$

$$\Rightarrow f_n - F_{n-1} - F_{n-2} = 0 \quad \text{for } n \geq 2 \rightarrow (1)$$

AE : $k^2 - k - 1 = 0$.

$$k = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \quad (\text{real } \& \text{ distinct roots})$$

$$k_1 = \frac{1+\sqrt{5}}{2}, \quad k_2 = \frac{1-\sqrt{5}}{2}$$

General soln of (1) is

$$F_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n \rightarrow (2)$$

, where A & B are arbitrary constants.

Given $F_0 = 0$, $f_1 = 1$.

sub $n=0$ in (2), $f_0 = A + B$

$$\boxed{0 = A + B} \rightarrow (3)$$

sub $n=1$ in (2), $f_1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$$\Rightarrow \boxed{1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)} \rightarrow (4)$$

Eqn (3) $\times \left(\frac{1+\sqrt{5}}{2}\right)$ gives $0 = \left(\frac{1+\sqrt{5}}{2}\right)A + \left(\frac{1+\sqrt{5}}{2}\right)B \rightarrow (5)$

(4) $-$ (5) $\Rightarrow$ $1 = B\left(\frac{1-\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$

$$1 = B\left(-\frac{2\sqrt{5}}{2}\right) \Rightarrow \boxed{B = -1/\sqrt{5}}$$

from (3), $A = -B \Rightarrow \boxed{A = 1/\sqrt{5}}$

sub in (2),

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$